

# TOPOLOGY AND GEOMETRY OF COHOMOLOGY JUMP LOCI

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**ABSTRACT.** We elucidate the key role played by formality in the theory of characteristic and resonance varieties. We define relative characteristic and resonance varieties,  $V_k$  and  $R_k$ , related to twisted group cohomology with coefficients of arbitrary rank. We show that the germs at the origin of  $V_k$  and  $R_k$  are analytically isomorphic, if the group is 1-formal; in particular, the tangent cone to  $V_k$  at 1 equals  $R_k$ . These new obstructions to 1-formality lead to a striking rationality property of the usual resonance varieties. A detailed analysis of the irreducible components of the tangent cone at 1 to the first characteristic variety yields powerful obstructions to realizing a finitely presented group as the fundamental group of a smooth, complex quasi-projective algebraic variety. This sheds new light on a classical problem of J.-P. Serre. Applications to arrangements, configuration spaces, coproducts of groups, and Artin groups are given.

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## 1. INTRODUCTION

A recurring theme in algebraic topology and geometry is the extent to which the cohomology of a space reflects the underlying topological or geometric properties of that space. In this paper, we focus on the degree-one cohomology jumping loci of the fundamental group  $G$  of a connected CW-complex  $M$ : the characteristic varieties  $\mathcal{V}_k(G)$ , the resonance varieties  $\mathcal{R}_k(G)$ , as well as relative versions of both. Our goal is to relate these two sets of varieties, and to better understand their structural properties, under certain naturally defined conditions on  $M$ . In turn, this analysis yields new and powerful obstructions for a finitely generated group  $G$  to be 1-formal, or realizable as the fundamental group of a smooth, complex quasi-projective variety.

**1.1. Characteristic varieties.** Let  $M$  be a connected CW-complex with finite 1-skeleton, and fundamental group  $G = \pi_1(M)$ . Let  $\rho: \mathbb{B} \rightarrow \mathrm{GL}(V)$  be a rational representation of the linear algebraic group  $\mathbb{B}$ . The *relative characteristic varieties* associated to these data are the jump loci for twisted cohomology of  $M$  with coefficients in  $V$ ,

$$(1.1) \quad \mathcal{V}_k^i(M, \rho) = \{\rho' \in \mathrm{Hom}_{\mathrm{groups}}(G, \mathbb{B}) \mid \dim_{\mathbb{C}} H^i(M, {}_{\rho\rho'}V) \geq k\},$$

where  ${}_{\rho\rho'}V$  is the  $G$ -module defined by the representation  $\rho \circ \rho': G \rightarrow \mathrm{GL}(V)$ . As long as the  $r$ -skeleton of  $M$  is finite, the sets  $\mathcal{V}_k^i(M, \rho)$  are Zariski closed subsets of the representation variety  $\mathrm{Hom}(G, \mathbb{B})$ , for all  $i \leq r$  and  $k \geq 1$ .

The usual characteristic varieties, denoted by  $\mathcal{V}_k^i(M)$ , correspond to the case when  $\mathbb{B} = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$  and  $\rho = \mathrm{id}_{\mathbb{B}}$ . These varieties emerged from the work of S. Novikov [61] on Morse theory for closed 1-forms on manifolds. Foundational results on the structure of the cohomology support loci for local systems on quasi-projective algebraic varieties were obtained by Beauville [5, 6], Green and Lazarsfeld [34], Simpson [72, 73], Arapura [3], and Campana [11].

Representation varieties have been intensively studied; see for instance [33, 41] and the references therein. It seems natural to begin a systematic investigation of their filtration by relative characteristic varieties. We consider here only the jump loci in degree  $i = 1$ .<sup>1</sup> These loci depend solely on  $G = \pi_1(M)$ , so we write them as  $\mathcal{V}_k(G, \rho) = \mathcal{V}_k^1(M, \rho)$ . We work with arbitrary representations  $\rho$ , since this general approach provides a clearer conceptual framework for our results. Moreover, as we note in Example 3.2, the relative characteristic varieties may well contain more information than the usual ones. When it comes to applications, though, we concentrate exclusively on the varieties  $\mathcal{V}_k(G) = \mathcal{V}_k(G, \mathrm{id}_{\mathbb{C}^*})$ , sitting inside the character variety  $\mathbb{T}_G = \mathrm{Hom}(G, \mathbb{C}^*)$ .

**1.2. Resonance varieties.** Keeping the notation from above, let  $\mathfrak{b}$  the Lie algebra of  $\mathbb{B}$ , and let  $\theta = d_1(\rho): \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  be the induced representation on tangent spaces. Denote by  $V \rtimes_{\theta} \mathfrak{b}$  the corresponding semidirect product, with Lie algebra structure defined by  $[(u, g), (v, h)] = (\theta(g)(v) - \theta(h)(u), [g, h])$ . Let  $H^{\bullet}M = H^{\bullet}(M, \mathbb{C})$  be the cohomology ring of  $M$ , and consider the graded Lie algebra

$$(1.2) \quad H^{\bullet}M \otimes (V \rtimes_{\theta} \mathfrak{b}),$$

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<sup>1</sup>The higher degree cohomology jumping loci will be treated in a forthcoming paper.

with Lie bracket given by  $[a \otimes y, b \otimes z] = ab \otimes [y, z]$ . Note that the Lie identities are satisfied for (1.2) only in the graded sense, according to the Koszul sign conventions; in particular, the Lie bracket is symmetric on degree one elements.

Now let  $x$  be an element in  $H^1 M \otimes \mathfrak{b}$  with the property that  $[x, x] = 0 \in H^2 M \otimes (V \rtimes_{\theta} \mathfrak{b})$ . As we show in §3.12, the restriction of the adjoint operator  $\text{ad}_x$  to the graded Lie ideal  $H^\bullet M \otimes V$  yields a cochain complex, called the *relative Aomoto complex* of  $H^\bullet M$  with respect to  $x$ ,

$$(1.3) \quad (H^\bullet M \otimes V, \text{ad}_x) : \cdots \longrightarrow H^i M \otimes V \xrightarrow{\text{ad}_x} H^{i+1} M \otimes V \longrightarrow \cdots.$$

The *relative resonance varieties* of  $M$ , associated to the representation  $\theta : \mathfrak{b} \rightarrow \mathfrak{gl}(V)$ , are the jumping loci for the homology of this cochain complex:

$$(1.4) \quad \mathcal{R}_k^i(M, \theta) = \{x \in H^1 M \otimes \mathfrak{b} \mid [x, x] = 0 \text{ and } \dim_{\mathbb{C}} H^i(H^\bullet M \otimes V, \text{ad}_x) \geq k\}.$$

The loci  $\mathcal{R}_k^1(M, \theta)$  depend only on  $G = \pi_1(M)$ , so we write them as  $\mathcal{R}_k(G, \theta)$ . It is readily seen that  $\mathcal{R}_k(G, \theta)$  is a homogeneous algebraic subvariety of the affine space  $H^1(G, \mathbb{C}) \otimes \mathfrak{b}$ , for each  $k \geq 1$ .

As mentioned before, we are mainly interested here in the case when  $\mathfrak{b} = \mathbb{C} = \mathfrak{gl}_1(\mathbb{C})$  and  $\theta = \text{id}_{\mathfrak{b}}$ . Then, as we show in Lemma 3.15, every element  $z \in H^1 M \cong H^1 M \otimes \mathfrak{b}$  satisfies  $[z, z] = 0$ , and (1.3) becomes the usual Aomoto complex,  $(H^\bullet M, \mu_z)$ , where  $\mu_z$  denotes left-multiplication by  $z$ . Consequently, the relative resonance varieties  $\mathcal{R}_k^i(M, \text{id}_{\mathbb{C}})$  coincide with the usual resonance varieties  $\mathcal{R}_k^i(M)$ , first defined by Falk [30] in the context of complex hyperplane arrangements, and further analyzed in [16, 55, 49, 31]. In the applications, we will focus on the varieties  $\mathcal{R}_k(G) := \mathcal{R}_k(G, \text{id}_{\mathbb{C}})$ , sitting inside the affine space  $H^1(G, \mathbb{C})$ .

**1.3. The tangent cone theorem.** The key topological property that allows us to relate the characteristic and resonance varieties of a space  $M$  is *formality*, in the sense of D. Sullivan [74]. Since we deal solely with the fundamental group  $G = \pi_1(M)$ , we only need  $G$  to be 1-*formal*. This property requires that  $E_G$ , the Malcev Lie algebra of  $G$  (over  $\mathbb{C}$ ), be isomorphic, as a filtered Lie algebra, to the holonomy Lie algebra  $\mathfrak{H}(G) = \mathbb{L}/\langle \text{im } \partial_G \rangle$ , completed with respect to bracket length filtration. Here  $\mathbb{L}$  denotes the free Lie algebra on  $H_1(G, \mathbb{C})$ , while  $\partial_G$  denotes the comultiplication map  $H_2(G, \mathbb{C}) \rightarrow \bigwedge^2 H_1(G, \mathbb{C})$ .

A group  $G$  is 1-formal if and only if  $E_G$  can be presented with quadratic relations only; see Section 2 for details. Many interesting groups fall in this class: knot groups [74], certain pure braid groups of Riemann surfaces [9, 37], pure welded braid groups [7], fundamental groups of compact Kähler manifolds [18], fundamental groups of complements of hypersurfaces in  $\mathbb{CP}^n$  [42], and finitely generated Artin groups [41] are all 1-formal.

Our starting point is a result of Kapovich–Millson [41], establishing an analytic isomorphism of germs of representation varieties,

$$(1.5) \quad (\text{Hom}_{\text{Lie}}(\mathfrak{H}(G), \mathfrak{b}), 0) \xrightarrow{\cong} (\text{Hom}_{\text{groups}}(G, \mathbb{B}), 1),$$

valid for all finitely generated, 1-formal groups  $G$ . As we note in Lemma 3.13, the representation space  $\text{Hom}_{\text{Lie}}(\mathfrak{H}(G), \mathfrak{b})$  is naturally identified with the quadratic cone  $\{x \in H^1 G \otimes \mathfrak{b} \mid [x, x] = 0\}$ . Our first main result reads as follows.

**Theorem A.** *Let  $G$  be a finitely generated, 1-formal group, and let  $\rho: \mathbb{B} \rightarrow \mathrm{GL}(V)$  be a rational representation of a linear algebraic group, with differential  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$ . Then, for each  $k \geq 1$ ,*

- (1) *The isomorphism (1.5) induces an analytic isomorphism of germs,*

$$(\mathcal{R}_k(G, \theta), 0) \xrightarrow{\cong} (\mathcal{V}_k(G, \rho), 1).$$

- (2) *This in turn induces an isomorphism between the tangent cone variety at the origin,  $TC_1(\mathcal{V}_k(G, \rho))$ , and the resonance variety  $\mathcal{R}_k(G, \theta)$ .*  
 (3) *When  $\mathbb{B} = \mathbb{C}^*$  and  $\rho = \mathrm{id}_{\mathbb{B}}$ , the isomorphism (1.5) is induced by the usual exponential map,  $\exp: \mathrm{Hom}(G, \mathbb{C}) \rightarrow \mathrm{Hom}(G, \mathbb{C}^*)$ .*

Theorem A sharpens and extends several results from [29, 70, 16], which only apply to the case when  $G$  is the fundamental group of the complement of a complex hyperplane arrangement. (Further information in the case of hypersurface arrangements can be found in [47, 22].) The point is that only a *topological* property—namely, 1-formality—is needed for the conclusion of Theorem A to hold. A similar approach (in terms of the relative Malcev completion, in the sense of Hain [36]) was used by Narkawicz [60] in the case of hyperplane arrangements.

For an arbitrary finitely presented group  $G$ , the tangent cone to  $\mathcal{V}_k(G)$  at the origin is contained in  $\mathcal{R}_k(G)$ , see Libgober [48]. Yet the inclusion may well be strict. In fact, as noted in Example 7.5 and Remark 10.3, the tangent cone formula from Theorem A fails even for fundamental groups of smooth, quasi-projective varieties.

Theorem A provides a new, and quite powerful obstruction to 1-formality of groups—and thus, to formality of spaces. As illustrated in Example 8.2, this obstruction complements, and in some cases strengthens, classical obstructions to (1-) formality, due to Sullivan, such as the existence of an isomorphism  $\mathrm{gr}(G) \otimes \mathbb{C} \cong \mathfrak{H}(G)$ .

**1.4. Further obstructions to 1-formality.** As a first application of Theorem A, we obtain the following result.

**Theorem B.** *Let  $G$  be a finitely generated, 1-formal group. Then:*

- (1) *All irreducible components of  $\mathcal{R}_k(G)$  are linear subspaces of  $H^1(G, \mathbb{C})$ , defined over  $\mathbb{Q}$ .*  
 (2) *All irreducible components of  $\mathcal{V}_k(G)$  containing 1 are connected subtori of the character variety  $\mathbb{T}_G = \mathrm{Hom}(G, \mathbb{C}^*)$ .*  
 (3) *Let  $\{\mathcal{V}_k^\alpha\}_\alpha$  be the irreducible components of  $\mathcal{V}_k(G)$  containing 1. Then their tangent spaces at the identity,  $\{T_1(\mathcal{V}_k^\alpha)\}_\alpha$ , are the irreducible components of  $\mathcal{R}_k(G)$ .*

The subtle linearity and rationality properties from (1) above reveal a striking phenomenon in non-simply connected rational homotopy theory: the existence of finite-dimensional, graded-commutative  $\mathbb{Q}$ -algebras, which are *not* realizable as cohomology rings of finite, formal CW-complexes; see Example 4.6. (By [74], this cannot happen in the 1-connected case.)

The Alexander invariant,  $B_G = G'/G''$ , viewed as a module over the group ring  $\mathbb{Z}G_{\mathrm{ab}}$ , is a classical object, extensively studied. In Theorem 5.6, we give an explicit description

of the  $I$ -adic completion of  $B_G \otimes \mathbb{Q}$ , involving only the rational holonomy Lie algebra of  $G$ . This description is valid for an arbitrary finitely generated, 1-formal group  $G$ .

**1.5. Serre's question.** As is well-known, every finitely presented group  $G$  is the fundamental group of a smooth, compact, connected 4-dimensional manifold  $M$ . Requiring a complex structure on  $M$  is no more restrictive, as long as one is willing to go up in dimension; see Taubes [75]. Requiring that  $M$  be a compact Kähler manifold, though, puts extremely strong restrictions on what  $G = \pi_1(M)$  can be. We refer to [2] for a comprehensive survey of Kähler groups.

J.-P. Serre asked the following question: which finitely presented groups can be realized as fundamental groups of smooth, connected, quasi-projective, complex algebraic varieties? Following Catanese [13], we shall call a group  $G$  arising in this fashion a *quasi-projective* group.

In this context, one may also consider the larger class of quasi-compact Kähler manifolds, of the form  $M = \overline{M} \setminus D$ , where  $\overline{M}$  is compact Kähler and  $D$  is a normal crossing divisor. If  $G = \pi_1(M)$  with  $M$  as above, we say  $G$  is a *quasi-Kähler* group.

The first obstruction to quasi-projectivity was given by J. Morgan: If  $G$  is a quasi-projective group, then  $E_G = \widehat{\mathbb{L}}/J$ , where  $\mathbb{L}$  is a free Lie algebra with generators in degrees 1 and 2, and  $J$  is a closed Lie ideal, generated in degrees 2, 3 and 4; see [59, Corollary 10.3]. By refining Morgan's techniques, Kapovich and Millson obtained analogous quasi-homogeneity restrictions, on certain singularities of representation varieties of  $G$  into reductive algebraic Lie groups; see [41, Theorem 1.13]. Another obstruction is due to Arapura: If  $G$  is quasi-Kähler, then the characteristic variety  $\mathcal{V}_1(G)$  is a union of (possibly translated) subtori of  $\mathbb{T}_G$ ; see [3, p. 564].

If the group  $G$  is 1-formal, then  $E_G = \widehat{\mathbb{L}}/J$ , with  $\mathbb{L}$  generated in degree 1 and  $J$  generated in degree 2; thus,  $G$  verifies Morgan's test. It is therefore natural to explore the relationship between 1-formality and quasi-projectivity. (In contrast with the Kähler case, it is known from [59, 41] that these two properties are independent.) Another motivation for our investigation comes from the study of fundamental groups of complements of plane algebraic curves. This class of 1-formal, quasi-projective groups contains, among others, the celebrated Stallings group; see [67].

**1.6. Position and resonance obstructions.** Our main contribution to Serre's problem is Theorem C below, which provides a new type of restriction on fundamental groups of smooth, quasi-projective complex algebraic varieties. In the presence of 1-formality, this restriction is expressed entirely in terms of a classical invariant, namely the cup-product map  $\cup_G: \bigwedge^2 H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C})$ .

**Theorem C.** *Let  $M$  be a connected, quasi-compact Kähler manifold. Set  $G = \pi_1(M)$  and let  $\{\mathcal{V}^\alpha\}$  be the irreducible components of  $\mathcal{V}_1(G)$  containing 1. Denote by  $\mathcal{T}^\alpha$  the tangent space at 1 of  $\mathcal{V}^\alpha$ . Then the following hold.*

- (1) *Every positive-dimensional tangent space  $\mathcal{T}^\alpha$  is a  $p$ -isotropic linear subspace of  $H^1(G, \mathbb{C})$ , of dimension at least  $2p + 2$ , for some  $p = p(\alpha) \in \{0, 1\}$ .*
- (2) *If  $\alpha \neq \beta$ , then  $\mathcal{T}^\alpha \cap \mathcal{T}^\beta = \{0\}$ .*

Assume further that  $G$  is 1-formal. Let  $\{\mathcal{R}^\alpha\}$  be the irreducible components of  $\mathcal{R}_1(G)$ . Then the following hold.

- (3) The collection  $\{\mathcal{T}^\alpha\}$  coincides with the collection  $\{\mathcal{R}^\alpha\}$ .
- (4) For  $1 \leq k \leq b_1(G)$ , we have  $\mathcal{R}_k(G) = \{0\} \cup \bigcup_\alpha \mathcal{R}^\alpha$ , where the union is over all components  $\mathcal{R}^\alpha$  such that  $\dim \mathcal{R}^\alpha > k + p(\alpha)$ .
- (5) The group  $G$  has a free quotient of rank at least 2 if and only if  $\mathcal{R}_1(G)$  strictly contains  $\{0\}$ .

Here, we say that a non-zero subspace  $V \subseteq H^1(G, \mathbb{C})$  is 0- (respectively, 1-) *isotropic* if the restriction of the cup-product map,  $\cup_G: \bigwedge^2 V \rightarrow \cup_G(\bigwedge^2 V)$ , is equivalent to  $\cup_C: \bigwedge^2 H^1(C, \mathbb{C}) \rightarrow H^2(C, \mathbb{C})$ , where  $C$  is a non-compact (respectively, compact) smooth, connected complex curve. See 6.5 for a more precise definition. The relation between Theorem C and the isotropic subspace theorems of Bauer [4] and Catanese [12] is discussed in detail in [21].

In this paper, we consider only components of  $\mathcal{V}_1(G)$  containing the origin. For a detailed analysis of translated components, we refer to [20] and [25].

Theorem C is derived from basic results of Arapura [3] on quasi-Kähler groups, plus our Theorem B, in the 1-formal case. Part (2) is a novel viewpoint, further pursued in [25] to obtain a completely new type of obstruction to quasi-projectivity. Part (3) follows from 1-formality alone, cf. Theorem B(3). As shown in Examples 6.18 and 8.1, the other four properties from Theorem C require in an essential way the quasi-Kähler hypothesis.

For an arrangement complement  $M$ , Parts (2) and (4) of the above theorem were proved by Libgober and Yuzvinsky in [49], using completely different methods.

The “position” obstructions (1) and (2) in Theorem C can be viewed as a much strengthened form of Arapura’s theorem: they give information on how the components of  $\mathcal{V}_1(G)$  passing through the origin intersect at that point, and how their tangent spaces at 1 are situated with respect to the cup-product map  $\cup_G$ .

We may also view conditions (1)–(5) as a set of “resonance” obstructions for a 1-formal group to be quasi-Kähler, or for a quasi-Kähler group to be 1-formal. Since the class of homotopy types of compact Kähler manifolds is strictly larger than the class of homotopy types of smooth projective varieties (see Voisin [77]), one may wonder whether the class of quasi-Kähler groups is also strictly larger than the class of quasi-projective groups.

**1.7. Applications.** In the last four sections, we illustrate the efficiency of our obstructions to 1-formality and quasi-Kählerianity with several classes of examples.

In Section 9, we consider wedges and products of spaces. Our analysis of resonance varieties of wedges, together with Theorem C, shows that 1-formality and quasi-Kählerianity behave quite differently with respect to the coproduct operation for groups. Indeed, if  $G_1$  and  $G_2$  are 1-formal, then  $G_1 * G_2$  is also 1-formal; but, if in addition both factors are non-trivial, presented by commutator relators only, and one of them is non-free, then  $G_1 * G_2$  is not quasi-Kähler. As a consequence of the position obstruction from Theorem C(1), we also show that the quasi-Kählerianity of  $G_1 * G_2$ , where the groups  $G_i$  are assumed only finitely presented with infinite abelianization, implies the vanishing of  $\cup_{G_1}$  and  $\cup_{G_2}$ .



When it comes to resonance varieties, real subspace arrangements offer a stark contrast to complex hyperplane arrangements, cf. [55, 56]. If  $M$  is the complement of an arrangement of transverse planes through the origin of  $\mathbb{R}^4$ , then  $G = \pi_1(M)$  passes Sullivan's gr-test. Yet, as we note in Section 8, the group  $G$  may fail the tangent cone formula from Theorem A, and thus be non-1-formal; or,  $G$  may be 1-formal, but fail tests (1), (2), (4) from Theorem C, and thus be non-quasi-Kähler.

In Section 10, we apply our techniques to the configuration spaces  $M_{g,n}$  of  $n$  ordered points on a closed Riemann surface of genus  $g$ . Clearly, the surface pure braid groups  $P_{g,n} = \pi_1(M_{g,n})$  are quasi-projective. On the other hand, if  $n \geq 3$ , the variety  $\mathcal{R}_1(P_{1,n})$  is irreducible and non-linear. Theorem C(3) shows that  $P_{1,n}$  is not 1-formal, thereby verifying a statement of Bezrukavnikov [9].

We conclude in Section 11 with a study of Artin groups associated to finite, labeled graphs, from the perspective of their cohomology jumping loci. As shown in [41], all Artin groups are 1-formal; thus, they satisfy Morgan's homogeneity test. Moreover, as we show in Proposition 11.5 (building on work from [66]), the first characteristic varieties of right-angled Artin groups are unions of coordinate subtori; thus, all such groups pass Arapura's  $\mathcal{V}_1$ -test.

In [41, Theorem 1.1], Kapovich and Millson establish, by a fairly involved argument, the existence of infinitely many Artin groups that are not realizable by smooth, quasi-projective varieties. Using the isotropicity obstruction from Theorem C(1), we show that a right-angled Artin group  $G_\Gamma$  is quasi-Kähler if and only if  $\Gamma$  is a complete, multi-partite graph, in which case  $G_\Gamma$  is actually quasi-projective. This result provides a complete—and quite satisfying—answer to Serre's problem within this class of groups. In the process, we take a first step towards solving the problem for all Artin groups, by answering it at the level of associated Malcev Lie algebras. We also determine precisely which right-angled Artin groups are Kähler.

The approach to Serre's problem taken in this paper—based on the obstructions from Theorem C—has led to complete answers for several other classes of groups:

- In [24], we classify the quasi-Kähler groups within the class of groups introduced by Bestvina and Brady in [8].
- In [25], we determine precisely which fundamental groups of boundary manifolds of line arrangements in  $\mathbb{CP}^2$  are quasi-projective groups.
- In [27], we decide which 3-manifold groups are Kähler groups, thus answering a question raised by S. Donaldson and W. Goldman in 1989.
- In [64], we show that the fundamental groups of a certain natural class of graph links in  $\mathbb{Z}$ -homology spheres are quasi-projective if and only if the corresponding links are Seifert links.

The computations from Section 11.4 have been pursued in [68], leading to a complete description of the characteristic varieties, in all degrees, for toric complexes associated to arbitrary finite simplicial complexes.

The obstructions from Theorem C are complemented by new methods of constructing interesting (quasi-)projective groups. These methods, developed in [24] and [26], lead to

a negative answer to the following question posed by J. Kollár in [43]: Is every projective group commensurable (up to finite kernels) with a group admitting a quasi-projective variety as classifying space?

## 2. HOLONOMY LIE ALGEBRA, MALCEV COMPLETION, AND 1-FORMALITY

Given a group  $G$ , there are several Lie algebras attached to it: the associated graded Lie algebra  $\mathrm{gr}^*(G)$ , the holonomy Lie algebra  $\mathfrak{H}(G)$ , and the Malcev Lie algebra  $E_G$ . In this section, we review the constructions of these Lie algebras, and the related notion of 1-formality, which will be crucial in what follows. We conclude with some relevant facts from the deformation theory of representations.

**2.1. Holonomy Lie algebras.** We start by recalling the definition of the holonomy Lie algebra, due to K.-T. Chen [15].

Let  $M$  be a connected CW-complex. Let  $\mathbb{k}$  be a field of characteristic 0. Denote by  $\mathbb{L}^*(H_1M)$  the free Lie algebra on  $H_1M = H_1(M, \mathbb{k})$ , graded by bracket length, and use the Lie bracket to identify  $H_1M \wedge H_1M$  with  $\mathbb{L}^2(H_1M)$ . Write  $\langle U \rangle$  for the Lie ideal spanned by a subset  $U \subset \mathbb{L}^*(H_1M)$ . Set

$$(2.1) \quad \mathfrak{H}^*(M) := \mathbb{L}^*(H_1M) / \langle \mathrm{im}(\partial_M: H_2M \rightarrow \mathbb{L}^2(H_1M)) \rangle,$$

where  $\partial_M$  is induced by the homology diagonal,  $H_2M \rightarrow H_2(M \times M)$ , via the Künneth decomposition. When  $M$  has finite 1-skeleton, the dual of  $\partial_M$  is the cup-product map  $\cup_M: H^1(M, \mathbb{k}) \wedge H^1(M, \mathbb{k}) \rightarrow H^2(M, \mathbb{k})$ . Note that  $\mathfrak{H}(M)$  is a quadratic Lie algebra, in that it has a presentation with generators in degree 1 and relations in degree 2 only. We call  $\mathfrak{H}(M)$  the *holonomy Lie algebra* of  $M$  (over the field  $\mathbb{k}$ ).

Now let  $G$  be a group, with Eilenberg-MacLane space  $K(G, 1)$ . Define

$$(2.2) \quad \mathfrak{H}(G) := \mathfrak{H}(K(G, 1)).$$

If  $M$  is a CW-complex with  $G = \pi_1(M)$ , and if  $f: M \rightarrow K(G, 1)$  is a classifying map, then  $f$  induces an isomorphism on  $H_1$  and an epimorphism on  $H_2$ . This implies that

$$(2.3) \quad \mathfrak{H}(G) = \mathfrak{H}(M).$$

**2.2. Malcev Lie algebras.** Next, we recall some useful facts about the functorial Malcev completion of a group, following Quillen [69, Appendix A].

A *Malcev Lie algebra* is a  $\mathbb{k}$ -Lie algebra  $E$ , endowed with a decreasing, complete filtration

$$E = F_1E \supset \cdots \supset F_qE \supset F_{q+1}E \supset \cdots,$$

by  $\mathbb{k}$ -vector subspaces satisfying  $[F_rE, F_sE] \subset F_{r+s}E$  for all  $r, s \geq 1$ , and with the property that the associated graded Lie algebra,  $\mathrm{gr}_F^*(E) = \bigoplus_{q \geq 1} F_qE / F_{q+1}E$ , is generated by  $\mathrm{gr}_F^1(E)$ . By completeness of the filtration, the Campbell-Hausdorff formula from local Lie theory

$$(2.4) \quad e \cdot f = e + f + \frac{1}{2}[e, f] + \frac{1}{12}([e, [e, f]] + [f, [f, e]]) + \cdots$$

endows the underlying set of  $E$  with a group structure, denoted by  $\exp(E)$ . Clearly,  $\exp(E)$  is a uniquely divisible group.



The lower central series of a Lie algebra  $L$  is defined inductively by  $\Gamma_1 L = L$  and  $\Gamma_{q+1} L = [L, \Gamma_q L]$ . If  $L$  is nilpotent, the lower central series filtration gives a canonical Malcev Lie structure on  $L$ .

For a group  $G$ , denote by  $\widehat{\mathfrak{H}}(G)$  the completion of the holonomy Lie algebra with respect to the degree filtration. It is readily checked that  $\widehat{\mathfrak{H}}(G)$ , together with the canonical filtration of the completion, is a Malcev Lie algebra with  $\mathrm{gr}_F^*(\widehat{\mathfrak{H}}(G)) = \mathfrak{H}^*(G)$ .

In [69], Quillen associates to  $G$ , in a functorial way, a Malcev Lie algebra  $E_G$  and a group homomorphism  $\kappa_G: G \rightarrow \exp(E_G)$ . A key property of the Malcev completion  $(E_G, \kappa_G)$  is that  $\kappa_G$  induces an isomorphism of graded  $\mathbb{k}$ -Lie algebras

$$(2.5) \quad \mathrm{gr}^*(\kappa_G): \mathrm{gr}^*(G) \otimes \mathbb{k} \xrightarrow{\cong} \mathrm{gr}_F^*(E_G).$$

Here,  $\mathrm{gr}^*(G) = \bigoplus_{q \geq 1} \Gamma_q G / \Gamma_{q+1} G$  is the graded Lie algebra associated to the lower central series,  $G = \Gamma_1 G \supset \cdots \supset \Gamma_q G \supset \Gamma_{q+1} G \supset \cdots$ , defined inductively by setting  $\Gamma_{q+1} G = (G, \Gamma_q G)$ , where  $(A, B)$  denotes the subgroup generated by all commutators  $(g, h) = ghg^{-1}h^{-1}$  with  $g \in A$  and  $h \in B$ , and with the Lie bracket on  $\mathrm{gr}^*(G)$  induced by the commutator map  $(\cdot, \cdot): G \times G \rightarrow G$ . Note that the Lie algebra  $\mathrm{gr}(G)$  is generated by  $G_{\mathrm{ab}} := \mathrm{gr}^1(G)$ , the abelianization of  $G$ .

If  $N$  is a nilpotent group,  $E_N$  is a nilpotent Lie algebra, and the Malcev filtration coincides with the lower central series filtration. Hence,  $\exp(E_N)$  is a nilpotent group. The Malcev completion  $\kappa_N: N \rightarrow \exp(E_N)$  is universal for homomorphisms of  $N$  into nilpotent, uniquely divisible groups, and has torsion kernel and cokernel.

**2.3. Formal spaces and 1-formal groups.** The important notion of formality of a space was introduced by D. Sullivan in [74]. Let  $M$  be a connected CW-complex. Roughly speaking,  $M$  is formal if the rational cohomology algebra of  $M$  determines the rational homotopy type of  $M$ . Many interesting spaces are formal: compact Kähler manifolds [18], homogeneous spaces of compact connected Lie groups with equal ranks [74]; products and wedges of formal spaces are again formal.

**Definition 2.4.** A finitely generated group  $G$  is *1-formal* if its Malcev Lie algebra,  $E_G$ , is isomorphic to the completion of its holonomy Lie algebra,  $\widehat{\mathfrak{H}}(G)$ , as filtered Lie algebras.

A fundamental result of Sullivan [74] states that  $\pi_1(M)$  is 1-formal whenever  $M$  is formal, with finite 1-skeleton. The converse is not true in general, though it holds under certain additional assumptions, see [66]. It is well-known that  $G$  is 1-formal if and only if  $E_G$  is isomorphic to the degree completion of a quadratic Lie algebra, as filtered Lie algebras; see for instance [2]. Here are some motivating examples.

**Example 2.5.** If  $M$  is obtained from a smooth, complex projective variety  $\overline{M}$  by deleting a subvariety  $D \subset \overline{M}$  with  $\mathrm{codim} D \geq 2$ , then  $\pi_1(M) = \pi_1(\overline{M})$ . Hence,  $\pi_1(M)$  is 1-formal, by [18].

**Example 2.6.** Let  $W_*(H^m(M, \mathbb{C}))$  be the Deligne weight filtration [17] on the cohomology with complex coefficients of a connected smooth quasi-projective variety  $M$ . It follows from a basic result of J. Morgan [59, Corollary 10.3] that  $\pi_1(M)$  is 1-formal if  $W_1(H^1(M, \mathbb{C})) =$

0. By [17, Corollary 3.2.17], this vanishing property holds whenever  $M$  admits a non-singular compactification with trivial first Betti number. As noted in [42], these two facts together establish the 1-formality of fundamental groups of complements of projective hypersurfaces.

**Example 2.7.** If  $b_1(G) = 0$ , it follows from [69] that  $E_G \cong \widehat{\mathfrak{H}}(G) = 0$ , therefore  $G$  is 1-formal. If  $G$  is finite, Serre showed in [71] that  $G$  is the fundamental group of a smooth complex projective variety.

**2.8. Deformation theory of representations.** We close this section by recalling from Goldman–Millson [33] several relevant facts in deformation theory, together with an application from Kapovich–Millson [41] to 1-formal groups.

The key point here is the description of the category of germs of  $\mathbb{C}$ -analytic varieties,  $(X, x)$ , by functors of Artin rings. Let  $(\widehat{\mathcal{O}}, \mathfrak{m})$  denote the complete local ring of  $(X, x)$ . The corresponding functor associates to an Artin local  $\mathbb{C}$ -algebra  $(A, \mathfrak{m})$  the set of  $A$ -points,  $F_{X,x}(A) = \text{Hom}_{\text{loc}}(\widehat{\mathcal{O}}, A)$ , where  $\text{Hom}_{\text{loc}}$  denotes local  $\mathbb{C}$ -algebra morphisms.

Let  $H$  be a  $\mathbb{C}$ -algebra and  $L$  a  $\mathbb{C}$ -Lie algebra. On the vector space  $H \otimes L$ , define a bracket by

$$(2.6) \quad [a \otimes x, b \otimes y] = ab \otimes [x, y], \text{ for } a, b \in H \text{ and } x, y \in L.$$

If  $H$  is a commutative algebra, this functorial construction produces a Lie algebra  $H \otimes L$ , whereas if  $H^\bullet$  is graded-commutative, it produces a graded Lie algebra  $H^\bullet \otimes L$  (for which the Lie identities hold up to the standard sign conventions).

Now let  $G$  be a discrete, finitely generated group, and  $\mathbb{B}$  a linear algebraic group, with Lie algebra  $\mathfrak{b}$ . Denote by  $\text{Rep}(G, \mathbb{B})$  the variety of group representations of  $G$  into  $\mathbb{B}$ , and by  $\text{Rep}(\mathfrak{H}(G), \mathfrak{b})$  the variety of Lie algebra homomorphisms from  $\mathfrak{H}(G)$  to  $\mathfrak{b}$ .

Denote by  $(X, x)$  the analytic germ of  $\text{Rep}(G, \mathbb{B})$  at the trivial representation 1, and by  $F = F_{X,x}$  the corresponding functor. Given an Artin ring  $(A, \mathfrak{m})$ , the Lie algebra  $\mathfrak{m} \otimes \mathfrak{b}$  is nilpotent, since  $\mathfrak{m}^q = 0$  for  $q \gg 0$ . It follows from [33, Theorem 4.3] that

$$(2.7) \quad F(A) = \text{Hom}_{\text{groups}}(G, \exp(\mathfrak{m} \otimes \mathfrak{b})).$$

Using this description, together with the universality property of the Malcev completion, Kapovich and Millson obtained in [41, Theorem 17.1] the following result.

**Theorem 2.9** ([41]). *If the group  $G$  is 1-formal, there is an analytic isomorphism,*

$$(\text{Rep}(\mathfrak{H}(G), \mathfrak{b}), 0) \xrightarrow{\cong} (\text{Rep}(G, \mathbb{B}), 1),$$

*natural in  $\mathbb{B}$ .*

**Remark 2.10.** Suppose  $\mathbb{B}$  is abelian, in which case  $\mathfrak{b}$  is also abelian. For a finitely generated group  $G$ , denote by  $G_{\text{abf}} = G_{\text{ab}} / \text{Tors}(G_{\text{ab}})$  its maximal torsion-free abelian quotient, and by  $n = \text{rank}(G_{\text{abf}})$  its first Betti number. We then have natural analytic isomorphisms,

$$\begin{aligned} (\text{Rep}(G, \mathbb{B}), 1) &\cong (\text{Rep}(G_{\text{abf}}, \mathbb{B}), 1) \cong (\mathbb{B}, 1)^n, \\ (\text{Rep}(\mathfrak{H}(G), \mathfrak{b}), 0) &\cong (\text{Rep}(\mathfrak{H}(G_{\text{abf}}), \mathfrak{b}), 0) \cong (\mathfrak{b}, 0)^n. \end{aligned}$$

It follows that the analytic isomorphism from Theorem 2.9 is induced by the local isomorphism  $\exp: (\mathfrak{b}, 0) \xrightarrow{\sim} (\mathbb{B}, 1)$ . When  $\mathbb{B} = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ , the map  $\exp: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^*, 1)$  is just the usual exponential.

**Remark 2.11.** When  $\mathbb{B} = \mathrm{GL}_r(\mathbb{C})$ , the usual matrix exponential,  $\exp: \mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C}) \rightarrow \mathrm{GL}_r(A)$ , induces a group isomorphism,

$$(2.8) \quad \exp: \exp(\mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C})) \xrightarrow{\sim} 1 + \mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C}),$$

where the right-hand side denotes the subgroup of  $\mathrm{GL}_r(A)$  consisting of matrices of the form  $\mathrm{id} + X$ , with all entries of  $X$  belonging to  $\mathfrak{m}$ .

### 3. GERMS OF COHOMOLOGY JUMP LOCI

In this section we prove Theorem A from the Introduction.

**3.1. Relative characteristic and resonance varieties.** Let  $M$  be a connected CW-complex with finite 1-skeleton. Set  $G = \pi_1(M)$ . Let  $\mathbb{B}$  be a complex linear algebraic group, and fix a rational representation,  $\rho: \mathbb{B} \rightarrow \mathrm{GL}(V)$ , where  $V$  is a complex vector space. Denote by  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  the morphism of Lie algebras induced by  $\rho$  on tangent spaces. Recall the relative characteristic varieties  $\mathcal{V}_k^i(M, \rho)$  defined in (1.1), and set

$$(3.1) \quad \mathcal{V}_k^i(G, \rho) := \mathcal{V}_k^i(K(G, 1), \rho).$$

By considering a classifying map  $M \rightarrow K(G, 1)$ , it is readily seen that  $\mathcal{V}_k^1(M, \rho) = \mathcal{V}_k^1(G, \rho)$ .

In general, the (germs of) relative characteristic varieties contain more information than the usual characteristic varieties,  $\mathcal{V}_k^i(G) = \mathcal{V}_k^i(G, \mathrm{id}_{\mathbb{C}^*})$ . For instance, if  $G$  is a finitely generated nilpotent group, then  $\mathcal{V}_k^i(G)$  depends only on  $k$  and the usual Betti number  $b_i(G)$ ; see [50]. On the other hand,  $\mathcal{V}_k^i(G, \rho)$  may depend on more subtle data. We illustrate this point—and the usefulness of the relative characteristic varieties—with a simple example, involving nilpotent groups.

**Example 3.2.** Let  $G = \langle x_1, x_2 \mid ((x_1, x_2), x_1), ((x_1, x_2), x_2) \rangle$  be the Heisenberg group, with abelianization  $G_{\mathrm{ab}} = \mathbb{Z}^2$ . It is readily seen that  $\mathcal{V}_k^1(G) = \mathcal{V}_k^1(\mathbb{Z}^2)$ : both are equal to  $\{1\}$  if  $k \leq 2$ , and are empty otherwise.

Now let  $\rho: \mathbb{B} = \mathrm{SL}(2, \mathbb{C}) \hookrightarrow \mathrm{GL}(2, \mathbb{C})$  be the inclusion map. It is not difficult to check that both (reduced) germs,  $(\mathrm{Rep}(G, \mathbb{B}), 1)$  and  $(\mathrm{Rep}(\mathbb{Z}^2, \mathbb{B}), 1)$ , are isomorphic to the germ at 0 of the variety of commuting pairs of matrices from  $\mathfrak{b} = \mathfrak{sl}(2, \mathbb{C})$ ; see [33, p. 89].

Direct computation shows that  $\mathcal{V}_3^1(\mathbb{Z}^2, \rho) = \{1\}$ . On the other hand, consider the embedding  $\iota: (\mathbb{C}^2, 0) \hookrightarrow (\mathrm{Rep}(G, \mathbb{B}), 1)$ , which, under the above isomorphism, sends  $(u, v)$  to the pair  $\left(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\right) \in \mathfrak{b}^2$ . Another direct computation (compare with [1]) shows that  $\iota((\mathbb{C}^2, 0)) \subset (\mathcal{V}_3^1(G, \rho), 1)$ . Hence,  $(\mathcal{V}_3^1(G, \rho), 1) \not\cong (\mathcal{V}_3^1(\mathbb{Z}^2, \rho), 1)$ .

Now let  $\mathfrak{H}$  be a complex, finitely generated Lie algebra, and let  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  be a complex, finite-dimensional representation of a complex, finite-dimensional Lie algebra  $\mathfrak{b}$ . Define the *relative resonance varieties* of  $\mathfrak{H}$  (with respect to  $\theta$ ) by

$$(3.2) \quad \mathcal{R}_k^i(\mathfrak{H}, \theta) = \{\theta' \in \mathrm{Rep}(\mathfrak{H}, \mathfrak{b}) \mid \dim_{\mathbb{C}} H^i(\mathfrak{H}, \theta\theta' V) \geq k\},$$

where Lie algebra cohomology is taken by viewing  $V$  as a left  $\mathfrak{H}$ -module via  $\theta\theta'$ . For each  $i$ , we have a descending filtration  $\{\mathcal{R}_k^i(\mathfrak{H}, \theta)\}_{k \geq 0}$  on  $\text{Rep}(\mathfrak{H}, \mathfrak{b})$ . Note that the sets  $\mathcal{R}_k(\mathfrak{H}, \theta) := \mathcal{R}_k^1(\mathfrak{H}, \theta)$  are closed subvarieties of  $\text{Rep}(\mathfrak{H}, \mathfrak{b})$ . As we shall see in Corollary 3.18, if  $\mathfrak{H} = \mathfrak{H}(G)$  is the holonomy Lie algebra of  $G$ , then  $\mathcal{R}_k(\mathfrak{H}, \theta) = \mathcal{R}_k(G, \theta)$ .

As before, write  $\mathcal{R}_k(\mathfrak{H}) := \mathcal{R}_k^1(\mathfrak{H}, \text{id}_{\mathfrak{gl}_1(\mathbb{C})})$ . If  $\mathfrak{H} = \mathfrak{H}(G)$ , this is a subvariety of  $\text{Rep}(\mathfrak{H}, \mathfrak{gl}_1(\mathbb{C})) = H^1(G, \mathbb{C})$ .

**3.3. Low-dimensional cohomology and representations.** We first recall some standard facts from [39, Chapters VI and VII].

Let  $G$  be a finitely generated group and  $\rho: \mathbb{B} \rightarrow \text{GL}(V)$  a rational representation. Denote by  $Z^i(G, \cdot)$  the  $i$ -cocycles for group cohomology, for  $i = 0, 1$ . Consider the semidirect product of groups  $V \rtimes_{\rho} \mathbb{B}$ , and the morphism of varieties

$$p: \text{Rep}(G, V \rtimes_{\rho} \mathbb{B}) \rightarrow \text{Rep}(G, \mathbb{B})$$

induced by the canonical projection,  $V \rtimes_{\rho} \mathbb{B} \rightarrow \mathbb{B}$ .

**Lemma 3.4.** *With notation as above, the fiber of  $p$  over  $\rho'$  is  $Z^1(G, {}_{\rho\rho'}V)$ .*

Consider the variety

$$\text{Rep}_{\rho}(G, \mathbb{B}) = \{(v, \rho') \in V \times \text{Rep}(G, \mathbb{B}) \mid \rho\rho'(g) \cdot v = v, \text{ for all } g \in G\},$$

and the morphism induced by the second-coordinate projection,

$$q: \text{Rep}_{\rho}(G, \mathbb{B}) \rightarrow \text{Rep}(G, \mathbb{B}).$$

**Lemma 3.5.** *The fiber of  $q$  over  $\rho'$  is  $Z^0(G, {}_{\rho\rho'}V)$ .*

Let  $\mathfrak{H}$  be a finitely generated Lie algebra and  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  a representation, with the properties described at the end of §3.1. Consider the semidirect product Lie algebra,  $V \rtimes_{\theta} \mathfrak{b}$ , and the morphism of varieties

$$\mathfrak{p}: \text{Rep}(\mathfrak{H}, V \rtimes_{\theta} \mathfrak{b}) \rightarrow \text{Rep}(\mathfrak{H}, \mathfrak{b})$$

induced by the canonical projection,  $V \rtimes_{\theta} \mathfrak{b} \rightarrow \mathfrak{b}$ .

**Lemma 3.6.** *The fiber of  $\mathfrak{p}$  over  $\theta'$  is  $Z^1(\mathfrak{H}, {}_{\theta\theta'}V)$ .*

Now consider the variety

$$\text{Rep}_{\theta}(\mathfrak{H}, \mathfrak{b}) = \{(v, \theta') \in V \times \text{Rep}(\mathfrak{H}, \mathfrak{b}) \mid \theta\theta'(h) \cdot v = 0, \text{ for all } h \in \mathfrak{H}\},$$

and the morphism induced by the second-coordinate projection,

$$\mathfrak{q}: \text{Rep}_{\theta}(\mathfrak{H}, \mathfrak{b}) \rightarrow \text{Rep}(\mathfrak{H}, \mathfrak{b}).$$

**Lemma 3.7.** *The fiber of  $\mathfrak{q}$  over  $\theta'$  is  $Z^0(\mathfrak{H}, {}_{\theta\theta'}V)$ .*

**Remark 3.8.** If  $\rho: \mathbb{B} \rightarrow \text{GL}(V)$  is a rational representation, and  $\theta = d_1(\rho): \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  is the induced representation on tangent spaces, then the Lie algebra of the algebraic group  $V \rtimes_{\rho} \mathbb{B}$  is  $V \rtimes_{\theta} \mathfrak{b}$ ; see for instance [40].

**3.9. Beginning of proof of Theorem A.** We need one more lemma.

**Lemma 3.10.** *Let  $(A, \mathfrak{m})$  be an Artin local algebra. Given  $v \in A^r$  and  $s \in \mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C})$ , the following are equivalent:*

- (1)  $\exp(s)^k \cdot v = v$ , for some  $k \in \mathbb{Z} \setminus \{0\}$ .
- (2)  $\exp(s) \cdot v = v$ .
- (3)  $s \cdot v = 0$ .

*Proof.* The implications (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (2) are clear.

Now suppose (2) holds. Then  $as \cdot v = 0$ , for some  $a \in 1 + \mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C}) \subseteq \mathrm{GL}_r(A)$ . Hence, (3) holds.

Finally, (1)  $\Rightarrow$  (3) follows from (2)  $\Rightarrow$  (3), applied to  $ks$ .  $\square$

**Theorem 3.11.** *Let  $G$  be a 1-formal group, and  $\rho: \mathbb{B} \rightarrow \mathrm{GL}(V)$  a rational representation. Set  $\theta = d_1(\rho): \mathfrak{b} \rightarrow \mathfrak{gl}(V)$ . Then, for each  $k \geq 1$ , the analytic isomorphism from Theorem 2.9 induces an analytic isomorphism of germs,*

$$(\mathcal{R}_k(\mathfrak{H}(G), \theta), 0) \xrightarrow{\cong} (\mathcal{V}_k(G, \rho), 1).$$

*Proof.* Assume in Theorem 2.9 that  $\theta' \in \mathrm{Rep}(\mathfrak{H}(G), \mathfrak{b})$  corresponds to  $\rho' \in \mathrm{Rep}(G, \mathbb{B})$ . We need to show that  $\dim_{\mathbb{C}} Z^i(G, \rho' V) = \dim_{\mathbb{C}} Z^i(\mathfrak{H}(G), \theta \theta' V)$ , for  $i = 0, 1$ . For  $i = 1$ , this follows from Lemmas 3.4 and 3.6, by using the naturality property from Theorem 2.9.

Denote by  $F_X$  the functor of Artin rings corresponding to the germ at  $(0, 1)$  of the variety  $\mathrm{Rep}_{\rho}(G, \mathbb{B})$ . Set  $r = \dim_{\mathbb{C}} V$ . It follows from §2.8 that

$$F_X(A) = \{(v, \rho'') \in \mathfrak{m}^{\times r} \times \mathrm{Rep}(G, \exp(\mathfrak{m} \otimes \mathfrak{b})) \mid (\mathrm{id} \otimes \theta) \rho''(g) \cdot v = v, \forall g \in G\},$$

where  $\mathrm{id} \otimes \theta: \exp(\mathfrak{m} \otimes \mathfrak{b}) \rightarrow \exp(\mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C}))$  is the group morphism induced by  $\theta$ . Recall from (2.8) that  $\exp(\mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C})) \cong 1 + \mathfrak{m} \otimes \mathfrak{gl}_r(\mathbb{C})$ . Using Lemma 3.10, the universality and torsion properties of the Malcev completion explained in §2.2, and the 1-formality of  $G$ , we may naturally identify  $F_X(A)$  with

$$\{(v, \theta'') \in \mathfrak{m}^{\times r} \times \mathrm{Rep}(\mathfrak{H}(G), \mathfrak{m} \otimes \mathfrak{b}) \mid (\mathrm{id} \otimes \theta) \theta''(h) \cdot v = 0, \forall h \in \mathfrak{H}(G)\}.$$

Plainly, this set coincides with  $F_Y(A)$ , where  $F_Y$  is the functor of Artin rings corresponding to the germ at  $(0, 0)$  of the variety  $\mathrm{Rep}_{\theta}(\mathfrak{H}(G), \mathfrak{b})$ . By Lemmas 3.5 and 3.7,  $\dim_{\mathbb{C}} Z^0(G, \rho' V) = \dim_{\mathbb{C}} Z^0(\mathfrak{H}(G), \theta \theta' V)$ , and we are done.  $\square$

**3.12. Resonance and Aomoto complexes.** We now relate Lie algebra representations to quadratic cones.

Let  $M$  be a connected CW-complex with finite 1-skeleton and fundamental group  $G = \pi_1(M)$ . Denote by  $H^{\bullet}M$  the cohomology ring of  $M$  with  $\mathbb{C}$  coefficients. Given a complex Lie algebra  $L$ , consider the graded Lie algebra  $H^{\bullet}M \otimes L$  constructed in (2.6). Define the associated *quadratic cone* by

$$(3.3) \quad \mathcal{Q}(M, L) = \{x \in H^1 M \otimes L \mid [x, x] = 0\}.$$

Clearly,  $\mathcal{Q}(M, L)$  depends only on  $G$ , so we denote it by  $\mathcal{Q}(G, L)$ .

**Lemma 3.13.** *Let  $G$  be a finitely generated group. The linear isomorphism between  $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{H}^1(G), L)$  and  $H^1G \otimes L$  gives a natural identification,*

$$\mathrm{Rep}(\mathfrak{H}(G), L) \cong \mathcal{Q}(G, L).$$

*Proof.* Plainly,  $\mathrm{Rep}(\mathfrak{H}(G), L) = \{f \in \mathrm{Hom}_{\mathbb{C}}(H_1G, L) \mid \beta \circ \wedge^2 f \circ \partial_G = 0\}$ , where  $\beta: L \wedge L \rightarrow L$  is the Lie bracket, and  $\partial_G: H_2G \rightarrow \wedge^2 H_1G$  is defined in (2.1). The element  $f$  is naturally identified with  $x \in H^1G \otimes L$ . We need to check that

$$(3.4) \quad \beta \circ \wedge^2 f \circ \partial_G = 0 \iff [x, x] = 0.$$

Pick a  $\mathbb{C}$ -basis  $\{e_i\}$  for  $H_1G$ , and denote by  $\{e_i^*\}$  the dual basis of  $H^1G$ . Let  $\{y_k\}$  and  $\{z_a\}$  be  $\mathbb{C}$ -bases for  $H_2G$  and  $L$ . Write  $x = \sum_{i,a} t_i^a e_i^* \otimes z_a$ , where  $f(e_i) = \sum_a t_i^a z_a$ . Also write  $\partial_G(y_k) = \frac{1}{2} \sum_{i,j} \mu_{ij}^k e_i \wedge e_j$ , and  $[z_a, z_b] = \sum_c s_{ab}^c z_c$ . Note that  $e_i^* \cup e_j^*(y_k) = \mu_{ij}^k$ . It is now straightforward to check that both conditions from (3.4) are equivalent to  $\sum_{i,j,a,b} t_i^a t_j^b \mu_{ij}^k s_{ab}^c = 0$ , for all  $k$  and  $c$ .  $\square$

Let  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  be a Lie algebra representation over  $\mathbb{C}$ , with  $\mathfrak{b}$  and  $V$  finite-dimensional. Tensoring with  $H^\bullet M$  the split exact sequence defined by the Lie semidirect product  $V \rtimes_\theta \mathfrak{b}$ , we obtain a graded vector space decomposition,

$$H^\bullet M \otimes (V \rtimes_\theta \mathfrak{b}) = (H^\bullet M \otimes V) \oplus (H^\bullet M \otimes \mathfrak{b}),$$

where  $H^\bullet M \otimes \mathfrak{b}$  is a Lie subalgebra, and  $H^\bullet M \otimes V$  is an abelian Lie ideal.

For  $x \in \mathcal{Q}(M, \mathfrak{b}) \subseteq H^1 M \otimes (V \rtimes_\theta \mathfrak{b})$ , we have  $0 = \mathrm{ad}_{[x,x]} = 2 \mathrm{ad}_x^2$ .

**Definition 3.14.** The *relative Aomoto complex* of  $H^\bullet M$  with respect to  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  and  $x \in \mathcal{Q}(M, \mathfrak{b})$  is the subcomplex  $(H^\bullet M \otimes V, \mathrm{ad}_x)$  of the cochain complex  $(H^\bullet M \otimes (V \rtimes_\theta \mathfrak{b}), \mathrm{ad}_x)$ .

The reason for this terminology is given by the next Lemma.

**Lemma 3.15.** *If  $\mathfrak{b} = \mathbb{C} = \mathfrak{gl}_1(\mathbb{C})$  and  $\rho = \mathrm{id}_{\mathfrak{b}}$ , then every element  $z \in H^1 M \cong H^1 M \otimes \mathfrak{b}$  satisfies  $[z, z] = 0$ , and the relative Aomoto complex  $(H^\bullet M \otimes V, \mathrm{ad}_{z \otimes 1})$  is identified with the usual Aomoto complex,  $(H^\bullet M, \mu_z)$ .*

*Proof.* The first assertion follows at once from (2.6), since  $\mathfrak{b}$  is abelian. To check the second claim, pick  $x = z \otimes 1 \in H^1 M \otimes \mathfrak{b} \cong H^1 M$ , and  $y \otimes 1 \in H^i M \otimes V \cong H^i M$ . Then

$$[x, y \otimes 1] = zy \otimes [(0, 1), (1, 0)] = z \cup y \otimes 1 \in H^{i+1} M \otimes V,$$

since the product is computed in the cohomology ring  $H^\bullet M$ , and the bracket in the Lie semidirect product  $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{C}$ . This identifies the relative Aomoto complex  $(H^\bullet M \otimes V, \mathrm{ad}_x)$  with the usual Aomoto complex  $(H^\bullet M, \mu_z)$ , and we are done.  $\square$

Returning now to the general situation, let  $G$  be a finitely generated group, and let  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  be a Lie algebra representation as above.

**Lemma 3.16.** *Given  $\theta' \in \mathrm{Rep}(\mathfrak{H}(G), \mathfrak{b})$ , we have a natural isomorphism*

$$H^1(\mathfrak{H}(G),_{\theta\theta'} V) \cong H^1(H^\bullet G \otimes V, \mathrm{ad}_x),$$

*where  $x \in \mathcal{Q}(G, \mathfrak{b})$  corresponds to  $\theta'$  under the isomorphism from Lemma 3.13.*



*Proof.* Denoting by  $Z^1$  and  $B^1$  the respective cocycles and coboundaries, we will prove the existence of natural compatible isomorphisms, for both of them.

For  $Z^1$ , it follows from Lemmas 3.6 and 3.13 that  $Z^1(\mathfrak{H}(G),_{\theta\theta'}V)$  is naturally identified with  $\{a \in H^1G \otimes V \mid [a+x, a+x] = 0\}$ . Clearly,  $[a+x, a+x] = 0$  if and only if  $[x, a] = 0$ , which happens precisely when  $a \in Z^1(H^\bullet G \otimes V, \text{ad}_x)$ .

Under this natural isomorphism,  $B^1(\mathfrak{H}(G),_{\theta\theta'}V)$  is identified with the image of the linear map,  $d: V \rightarrow \text{Hom}_{\mathbb{C}}(H_1G, V)$ , defined by  $dv(h) = \theta\theta'(h) \cdot v$ , for  $v \in V$  and  $h \in H_1G = \mathfrak{H}^1(G)$ . To finish the proof, we will show that  $d \equiv \text{ad}_x: H^0G \otimes V \rightarrow H^1G \otimes V$ .

As in the proof of Lemma 3.13, write  $x = \sum_i e_i^* \otimes b_i$ , with  $b_i = \theta'(e_i)$ . Then

$$[x, 1 \otimes v] = \sum_i e_i^* \otimes [b_i, v] = \sum_i e_i^* \otimes \theta(b_i) \cdot v = \sum_i e_i^* \otimes \theta\theta'(e_i) \cdot v = dv,$$

as claimed.  $\square$

**Remark 3.17.** One may generalize Definition 3.14 to an arbitrary connected, graded-commutative  $\mathbb{C}$ -algebra  $H^\bullet$ , by setting  $\mathcal{Q}(H^\bullet, \mathfrak{b}) = \{x \in H^1 \otimes \mathfrak{b} \mid [x, x] = 0\}$ , and proceeding as above. If  $H^\bullet \rightarrow H'^\bullet$  is a morphism of graded  $\mathbb{C}$ -algebras, which is an isomorphism in degree  $\bullet = 1$  and a monomorphism for  $\bullet = 2$ , then the corresponding quadratic cones are identified, as well as the cohomology in degree one of the associated Aomoto complexes.

Let  $H^1$  and  $H^2$  be  $\mathbb{C}$ -vector spaces, with  $\dim_{\mathbb{C}} H^1 < \infty$ , and let  $\mu: H^1 \wedge H^1 \rightarrow H^2$  be a linear map. Set  $H^0 = \mathbb{C} \cdot 1$ . Associated to these data, there is a connected, graded-commutative algebra,  $H^\bullet(\mu) := H^0 \oplus H^1 \oplus H^2$ . Define

$$\mathcal{R}_k(\mu, \theta) = \{x \in H^1 \otimes \mathfrak{b} \mid [x, x] = 0 \text{ and } \dim_{\mathbb{C}} H^1(H^\bullet(\mu) \otimes V, \text{ad}_x) \geq k\}.$$

Then the varieties  $\{\mathcal{R}_k(\mu, \theta)\}_{k \geq 0}$  depend only on the corestriction of  $\mu$  to its image, and the representation  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$ . As in §3.1, set  $\mathcal{R}_k(\mu) = \mathcal{R}_k(\mu, \text{id}_{\mathfrak{gl}(\mathbb{C})})$ .

Now let  $M$  be connected complex with finite 1-skeleton. Note that the degree-2 truncation of the cohomology ring,  $H^{\leq 2}(M, \mathbb{C})$ , is isomorphic to  $H^\bullet(\cup_M)$ . Thus, the varieties  $\mathcal{R}_k(\cup_M, \theta)$  coincide with the relative resonance varieties  $\mathcal{R}_k^1(M, \theta)$  defined in (1.4).

**Corollary 3.18.** *Let  $G$  be a finitely generated group and let  $\theta: \mathfrak{b} \rightarrow \mathfrak{gl}(V)$  be a morphism of complex Lie algebras, with  $\mathfrak{b}$  and  $V$  finite-dimensional. Then the varieties  $\mathcal{R}_k(\mathfrak{H}(G), \theta)$  and  $\mathcal{R}_k(G, \theta)$  are naturally isomorphic, for all  $k \geq 1$ .*

**3.19. End of proof of Theorem A.** The first part follows at once from Theorem 3.11 and Corollary 3.18.

For basic facts on tangent cones we refer the reader to Whitney's book [78]. By Remark 3.17, we infer that the isomorphism from Theorem 3.11 induces an isomorphism of varieties,  $TC_1(\mathcal{V}_k(G, \rho)) \cong TC_0(\mathcal{R}_k(\mu_G, \theta))$ , where  $\mu_G$  denotes the corestriction of  $\cup_G$  to its image. Note that  $\dim_{\mathbb{C}} H^\bullet(\mu_G) < \infty$ , and the (finite) matrices of the differential  $\text{ad}_x$  of the Aomoto complex  $(H^\bullet(\mu_G) \otimes V, \text{ad}_x)$  have entries consisting of linear forms. By construction,  $\mathcal{R}_k(\mu_G, \theta)$  is a cone, hence  $TC_0(\mathcal{R}_k(\mu_G, \theta)) \cong \mathcal{R}_k(\mu_G, \theta) \cong \mathcal{R}_k(G, \theta)$ . This proves the second part of Theorem A.

Finally, assume  $\mathbb{B} = \text{GL}_1(\mathbb{C}) = \mathbb{C}^*$  and  $\rho = \text{id}_{\mathbb{B}}$ . As noted in Remark 2.10, the local isomorphism  $(\text{Rep}(\mathfrak{H}(G), \mathbb{C}), 0) \cong (\text{Rep}(G, \mathbb{C}^*), 1)$  is induced by the usual exponential,

$$\exp: (\text{Hom}_{\text{groups}}(G_{\text{ab}}, \mathbb{C}), 0) \xrightarrow{\cong} (\text{Hom}_{\text{groups}}(G_{\text{ab}}, \mathbb{C}^*), 1),$$

which identifies both tangent spaces with  $H^1 G = H^1(G, \mathbb{C})$ . The proof of Theorem A is complete.

#### 4. 1-FORMALITY AND RATIONALITY PROPERTIES

In this section,  $G$  is a finitely generated group. We deduce from Theorem A two remarkable linearity and rationality properties of the resonance varieties  $\mathcal{R}_k(G)$ , in the presence of 1-formality.

**4.1. Structure of exponential tangent cones.** As usual, let  $\mathbb{T}_G = H^1(G, \mathbb{C}^*) = \text{Hom}(G, \mathbb{C}^*)$  be the character group of  $G$ . Consider the exponential homomorphism,  $\exp: H^1(G, \mathbb{C}) \rightarrow \mathbb{T}_G$ , with image  $\mathbb{T}_G^0$ , the connected component of the identity  $1 \in \mathbb{T}_G$ .

**Definition 4.2.** For a Zariski closed subset  $W \subseteq \mathbb{T}_G$ , define the *exponential tangent cone* of  $W$  at 1 by

$$\tau_1(W) = \{z \in H^1(G, \mathbb{C}) \mid \exp(tz) \in W, \text{ for all } t \in \mathbb{C}\}.$$

Plainly,  $\tau_1(W)$  depends only on the analytic germ of  $W$  at 1. It is also easy to check that  $\tau_1(W) \subseteq TC_1(W)$ , whence our terminology. The next Lemma is the exponential version of a result of Laurent [44, Lemme 3].

**Lemma 4.3.** *For any  $W$  as above,  $\tau_1(W)$  is a finite union of rationally defined linear subspaces of  $H^1(G, \mathbb{C})$ .*

*Proof.* Set  $n = b_1(G)$ . The coordinate ring of the complex torus  $\mathbb{T}_G^0 = (\mathbb{C}^*)^n$  is the Laurent polynomial ring in  $n$  variables,  $\mathbb{C}\mathbb{Z}^n = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ . It is enough to verify our claim for  $W = V(f)$ , where  $f = \sum_{u \in S} c_u t_1^{u_1} \cdots t_n^{u_n}$ , the support  $S \subseteq \mathbb{Z}^n$  is finite, and  $c_u \neq 0$ , for all  $u \in S$ . For a fixed  $z \in \mathbb{C}^n$ ,  $z \in \tau_1(W)$  if and only if the analytic function

$$(4.1) \quad \sum_{u \in S} c_u e^{\langle u, z \rangle t}$$

vanishes identically in  $t$ .

In turn, this condition is easy to check, by using the well-known fact that the exponential functions  $e^{ty_1}, \dots, e^{ty_r}$  are linearly independent, provided  $y_1, \dots, y_r$  are all distinct. Define  $\mathcal{P}$  to be the set of partitions  $p = S_1 \amalg \cdots \amalg S_k$  of  $S$ , having the property that  $\sum_{u \in S_i} c_u = 0$ , for  $i = 1, \dots, k$ . For  $p \in \mathcal{P}$ , define the rational linear subspace  $L(p) \subseteq \mathbb{C}^n$  by

$$L(p) = \{z \in \mathbb{C}^n \mid \langle u - v, z \rangle = 0, \forall u, v \in S_i, \forall 1 \leq i \leq k\}.$$

It is straightforward to check that  $\tau_1(W) = \bigcup_{p \in \mathcal{P}} L(p)$ , by grouping terms in (4.1).  $\square$

**4.4. Proof of Theorem B.** Let  $G$  be a (finitely generated) 1-formal group.

Part (1). Theorem A guarantees that  $\tau_1(\mathcal{V}_k(G)) = \mathcal{R}_k(G)$ . By Lemma 4.3, all irreducible components of  $\mathcal{R}_k(G)$  are rationally defined linear subspaces of  $H^1(G, \mathbb{C})$ .

Part (2). Let  $\mathcal{R}_k(G) = \bigcup_a L_a$  be the irreducible decomposition from Part (1). Then  $\exp(L_a)$  is a connected subtorus of  $\mathbb{T}_G$ , for all  $a$ , and  $T_1(\exp(L_a)) = L_a$ . Now let  $\{\mathcal{V}_k^\alpha\}_\alpha$  be the irreducible components of  $\mathcal{V}_k(G)$  containing 1. By Theorem A,  $\bigcup_\alpha \mathcal{V}_k^\alpha = \bigcup_a \exp(L_a)$ , near 1. Hence, each component  $\mathcal{V}_k^\alpha(G)$  is a connected subtorus of  $\mathbb{T}_G$ .

Part (3). Using again Theorem A, we deduce that  $\mathcal{R}_k(G) = TC_1(\mathcal{V}_k(G)) = \bigcup_{\alpha} T_1(\mathcal{V}_k^{\alpha})$ . By [40, 13.1], this gives the decomposition into irreducible components of  $\mathcal{R}_k(G)$ .

**4.5. Formal realizability of cohomology rings.** Theorem B indicates that the 1-formality property of a group imposes severe restrictions on its resonance varieties in degree one.

**Example 4.6.** Let  $K$  be the finite, 2-dimensional CW-complex associated to the following group presentation, with commutator-relators:

$$G = \langle x_1, x_2, x_3, x_4 \mid (x_1, x_2), (x_1, x_4) \cdot (x_2^{-2}, x_3), (x_1^{-1}, x_3) \cdot (x_2, x_4) \rangle.$$

The dual of the cup-product,  $\partial_K: H_2(K, \mathbb{C}) \rightarrow \bigwedge^2 H_1(K, \mathbb{C})$ , is given by

$$(4.2) \quad \begin{cases} \partial_K f_1 &= e_1 \wedge e_2, \\ \partial_K f_2 &= e_1 \wedge e_4 - 2e_2 \wedge e_3, \\ \partial_K f_3 &= -e_1 \wedge e_3 + e_2 \wedge e_4, \end{cases}$$

where  $\{e_i\}$  are the 1-cells of  $K$ , and  $\{f_j\}$  the 2-cells. It follows from (4.2) that  $\mathcal{R}_1(K) = \mathcal{R}_1(G)$  is given by the equation  $x_1^2 - 2x_2^2 = 0$ , where  $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$  are the coordinates corresponding to the canonical  $\mathbb{Q}$ -structure of  $H^1(K, \mathbb{C}) = H^1(G, \mathbb{C})$ .

Plainly, the rationality property from Theorem B(1) is violated by the resonance variety  $\mathcal{R}_1(G)$  from the previous example. This leads to the following corollary.

**Corollary 4.7.** *Let  $K$  be the CW-complex defined above. There is no formal CW-complex  $M$  with finite 1-skeleton such that  $H^{\leq 2}(M, \mathbb{Q}) \cong H^{\leq 2}(K, \mathbb{Q})$ , as graded rings.*

This corollary is in marked contrast with the following basic result in simply-connected rational homotopy theory.

**Theorem 4.8** (Sullivan [74]). *Let  $H^{\bullet}$  be a connected, finite-dimensional, graded-commutative algebra over  $\mathbb{Q}$ . If  $H^1 = 0$ , there is a 1-connected, finite, formal CW-complex  $M$ , such that  $H^{\bullet}(M, \mathbb{Q}) \cong H^{\bullet}$ , as graded  $\mathbb{Q}$ -algebras.*

## 5. ALEXANDER INVARIANTS OF 1-FORMAL GROUPS

Our goal in this section is to derive a relation between the Alexander invariant and the holonomy Lie algebra of a finitely generated, 1-formal group, over a characteristic zero field  $\mathbb{k}$ .

**5.1. Alexander invariants.** Let  $G$  be a group. Consider the exact sequence

$$(5.1) \quad 0 \longrightarrow G'/G'' \xrightarrow{j} G/G'' \xrightarrow{p} G_{\text{ab}} \longrightarrow 0,$$

where  $G' = (G, G)$ ,  $G'' = (G', G')$  and  $G_{\text{ab}} = G/G'$ . Conjugation in  $G/G''$  naturally makes  $G'/G''$  into a module over the group ring  $\mathbb{Z}G_{\text{ab}}$ . Following Fox [32], we call this module,

$$B_G = G'/G'',$$

the *Alexander invariant* of  $G$ . If  $G = \pi_1(M)$ , where  $M$  is a connected CW-complex, one has the following useful topological interpretation for the Alexander invariant. Let

$M' \rightarrow M$  be the Galois cover corresponding to  $G' \subset G$ . Then  $B_G \otimes \mathbb{k} = H_1(M', \mathbb{k})$ , and the action of  $G_{\text{ab}}$  corresponds to the action in homology of the group of covering transformations.

Now assume the group  $G$  is finitely generated. Then  $B_G \otimes \mathbb{k}$  is a finitely generated module over the Noetherian ring  $\mathbb{k}G_{\text{ab}}$ . Denote by  $I \subset \mathbb{k}G_{\text{ab}}$  the augmentation ideal and set  $X := G_{\text{ab}} \otimes \mathbb{k}$ . The  $I$ -adic completion  $\widehat{B_G \otimes \mathbb{k}}$  is a finitely generated module over  $\widehat{\mathbb{k}G_{\text{ab}}} \cong \mathbb{k}[[X]]$ , the formal power series ring on  $X$ . Note that  $\mathbb{k}[[X]]$  is also the  $(X)$ -adic completion of the polynomial ring  $\mathbb{k}[X]$ .

The above identification is induced by the  $\mathbb{k}$ -algebra map

$$(5.2) \quad \exp: \mathbb{k}G_{\text{ab}} \rightarrow \mathbb{k}[[X]],$$

defined by  $\exp(a) = e^{a \otimes 1}$ , for  $a \in G_{\text{ab}}$ . After completion, we obtain an isomorphism of filtered  $\mathbb{k}$ -algebras,  $\widehat{\exp}: \widehat{\mathbb{k}G_{\text{ab}}} \xrightarrow{\sim} \mathbb{k}[[X]]$ .

Another invariant associated to a finitely generated group  $G$  is the *infinitesimal Alexander invariant*,  $B_{\mathfrak{H}(G)}$ . By definition, this is the finitely generated graded module over  $\mathbb{k}[X]$  with presentation matrix

$$(5.3) \quad \nabla := \delta_3 + \text{id} \otimes \partial_G: \mathbb{k}[X] \otimes \left( \bigwedge^3 X \oplus Y \right) \rightarrow \mathbb{k}[X] \otimes \bigwedge^2 X,$$

where  $Y = H_2(G, \mathbb{k})$  and  $\delta_3(x \wedge y \wedge z) = x \otimes y \wedge z - y \otimes x \wedge z + z \otimes x \wedge y$ ; see [65, Theorem 6.2] for details on degrees. As the notation indicates, the graded module  $B_{\mathfrak{H}(G)}$  depends only on the holonomy Lie algebra of  $G$ .

**5.2. Alexander invariant and Malcev completion.** Let  $G$  be a finitely generated group, with Malcev Lie algebra  $E = E_G$ . We have an exact sequence of complete Lie algebras,

$$(5.4) \quad 0 \longrightarrow \overline{E'}/\overline{E''} \xrightarrow{\iota} E/\overline{E''} \xrightarrow{\pi} E/\overline{E'} \cong X \longrightarrow 0,$$

with  $E/\overline{E'} \cong X$  and  $\overline{E'}/\overline{E''}$  abelian. Here  $\overline{(\cdot)}$  denotes closure with respect to the topology defined by the filtration. Both  $E/\overline{E''}$  and  $E/\overline{E'}$  are endowed with the quotient filtrations induced from  $E$ , and  $\overline{E'}/\overline{E''}$  carries the subspace filtration induced from  $E/\overline{E''}$ . The adjoint action of  $X$  on  $E/\overline{E''}$  induces a  $\mathbb{k}[[X]]$ -module structure on  $\overline{E'}/\overline{E''}$ .

Theorem 3.5 from [65] gives a commutative diagram of groups,

$$(5.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & G'/G'' & \xrightarrow{j} & G/G'' & \xrightarrow{p} & G_{\text{ab}} \longrightarrow 0 \\ & & \downarrow \kappa_0 & & \downarrow \kappa'' & & \downarrow \kappa' \\ 0 & \longrightarrow & \exp(\overline{E'}/\overline{E''}) & \xrightarrow{\iota} & \exp(E/\overline{E''}) & \xrightarrow{\pi} & \exp(X) \longrightarrow 0 \end{array}$$

where both  $\kappa'$  and  $\kappa''$  are Malcev completions.

**Lemma 5.3.** *The map  $\kappa_0 \otimes \mathbb{k}: (G'/G'') \otimes \mathbb{k} \rightarrow \overline{E'}/\overline{E''}$  is exp-linear, that is,*

$$(\kappa_0 \otimes \mathbb{k})(\alpha \cdot \beta) = \exp(\alpha) \cdot (\kappa_0 \otimes \mathbb{k})(\beta),$$

for  $\alpha \in \mathbb{k}G_{\text{ab}}$  and  $\beta \in (G'/G'') \otimes \mathbb{k}$ .

*Proof.* It is enough to show that  $\kappa''(a \cdot j(b) \cdot a^{-1}) = e^{p(a) \otimes 1} \cdot \kappa''(j(b))$  for  $a \in G/G''$  and  $b \in G'/G''$ . To check this equality, recall the well-known conjugation formula in exponential groups (see Lazard [45]), which in our situation says

$$xyx^{-1} = \exp(\text{ad}_x)(y),$$

for  $x, y \in \exp(E/\overline{E''})$ . Hence  $\kappa''(a \cdot j(b) \cdot a^{-1}) = e^{\text{ad}_{\kappa''(a)}(\kappa''(j(b)))}$ , which equals  $e^{p(a) \otimes 1} \cdot \kappa''(j(b))$ , since  $\kappa' \circ p(a) = p(a) \otimes 1$ .  $\square$

Recall that  $B_G \otimes \mathbb{k}$  is a module over  $\mathbb{k}G_{\text{ab}}$ , and  $\overline{E'}/\overline{E''}$  is a module over  $\widehat{\mathbb{k}[X]} = \mathbb{k}[[X]]$ . Shift the Malcev filtration on  $\overline{E'}/\overline{E''}$  by setting  $F'_q \overline{E'}/\overline{E''} := F_{q+2} \overline{E'}/\overline{E''}$ , for each  $q \geq 0$ .

**Proposition 5.4.** *The  $\mathbb{k}$ -linear map  $\kappa_0 \otimes \mathbb{k}: B_G \otimes \mathbb{k} \rightarrow \overline{E'}/\overline{E''}$  induces a filtered  $\widehat{\text{exp}}$ -linear isomorphism between  $\widehat{B_G \otimes \mathbb{k}}$ , endowed with the filtration coming from the  $I$ -adic completion, and  $\overline{E'}/\overline{E''}$ , endowed with the shifted Malcev filtration  $F'$ .*

*Proof.* We start by proving that

$$(5.6) \quad \kappa_0 \otimes \mathbb{k} (I^q B_G \otimes \mathbb{k}) \subset F'_q \overline{E'}/\overline{E''}$$

for all  $q \geq 0$ . First note that  $\overline{E'}/\overline{E''} = F_2 \overline{E'}/\overline{E''}$ , by filtered exactness of (5.4), together with (2.5). Next, recall that  $\exp(I) \subset (X)$ . Finally, note that  $(X)^r F_s \overline{E'}/\overline{E''} \subset F_{r+s} \overline{E'}/\overline{E''}$  for all  $r, s$ . These observations, together with the  $\text{exp}$ -equivariance property from Lemma 5.3, establish the claim.

In view of (5.6),  $\kappa_0 \otimes \mathbb{k}$  induces a filtered,  $\widehat{\text{exp}}$ -linear map from  $\widehat{B_G \otimes \mathbb{k}}$  to  $\overline{E'}/\overline{E''}$ . We are left with checking this map is a filtered isomorphism. For that, it is enough to show

$$(5.7) \quad \text{gr}^q(\kappa_0 \otimes \mathbb{k}): \text{gr}_I^q(B_G \otimes \mathbb{k}) \longrightarrow \text{gr}_F^{q+2}(\overline{E'}/\overline{E''})$$

is an isomorphism, for each  $q \geq 0$ .

Recall from (5.5) that  $\kappa'' \circ j = \iota \circ \kappa_0$ . By a result of W. Massey [53, pp. 400–401], the map  $j: G'/G'' \rightarrow G/G''$  induces isomorphisms

$$(5.8) \quad \text{gr}^q(j) \otimes \mathbb{k}: \text{gr}_I^q(B_G \otimes \mathbb{k}) \xrightarrow{\cong} \text{gr}^{q+2}(G/G'') \otimes \mathbb{k}$$

for all  $q \geq 0$ . Since  $\kappa''$  is a Malcev completion, it induces isomorphisms

$$(5.9) \quad \text{gr}^q(\kappa''): \text{gr}^q(G/G'') \otimes \mathbb{k} \xrightarrow{\cong} \text{gr}_F^q(E/\overline{E''})$$

for all  $q \geq 1$ . Finally, the inclusion map  $\iota$  induces isomorphisms  $\text{gr}_F^q(\iota)$ , for all  $q \geq 2$ , again by filtered exactness of (5.4), since  $X$  is abelian. This finishes the proof.  $\square$

**5.5. Alexander invariants of 1-formal groups.** The next result is a new, explicit manifestation of D. Sullivan's general philosophy, according to which the algebraic topology of a formal space in characteristic zero is determined by the cohomology ring.

**Theorem 5.6.** *Let  $G$  be a 1-formal group. Then the  $I$ -adic completion of the Alexander invariant,  $\widehat{B_G \otimes \mathbb{k}}$ , is isomorphic to the  $(X)$ -adic completion of the infinitesimal Alexander invariant,  $\widehat{B_{S(G)}}$ , by a filtered  $\widehat{\text{exp}}$ -linear isomorphism.*

*Proof.* Set  $\mathfrak{H} := \mathfrak{H}(G)$ . Consider the following sequence of Lie algebras,

$$0 \longrightarrow \mathfrak{H}'/\mathfrak{H}'' \longrightarrow \mathfrak{H}/\mathfrak{H}'' \longrightarrow \mathfrak{H}/\mathfrak{H}' \longrightarrow 0.$$

Clearly,  $\mathfrak{H}/\mathfrak{H}'$  and  $\mathfrak{H}'/\mathfrak{H}''$  are abelian, and  $\mathfrak{H}/\mathfrak{H}' \cong X$ . Moreover, the sequence is exact in each degree.

Assign degree  $q$  to  $\bigwedge^q X$  and degree 2 to  $Y$  in (5.3). Then  $\mathfrak{H}'/\mathfrak{H}''$ , viewed as a graded  $\mathbb{k}[X]$ -module via the above exact sequence, is graded isomorphic to the  $\mathbb{k}[X]$ -module  $\text{coker}(\nabla)$ , see [65, Theorem 6.2]. Taking  $(X)$ -adic completions, the claim follows from Proposition 5.4, since  $\overline{E'}/\overline{E''} \cong \widehat{\mathfrak{H}'/\mathfrak{H}''}$ , by 1-formality.  $\square$

**Remark 5.7.** An alternative proof of Theorem A, in the case  $\rho = \text{id}_{\text{GL}_1(\mathbb{C})}$ , based on Theorem 5.6, goes as follows. Consider the Alexander invariant,  $B_G \otimes \mathbb{C}$ , over  $\mathbb{C}G_{\text{ab}}$ , and the infinitesimal Alexander invariant,  $B_{\mathfrak{H}(G)}$ , over  $\mathbb{C}[G_{\text{ab}} \otimes \mathbb{C}]$ . The varieties defined by the Fitting ideals of these two modules coincide, away from the origin, with  $\mathcal{V}_k(G)$ , and  $\mathcal{R}_k(G)$  respectively. By base change and Theorem 5.6, the Fitting ideals are identified in the 1-formal case via  $\widehat{\text{exp}}$ , upon completion. Since  $\mathbb{C}[[X]]$  is faithfully flat over  $\mathbb{C}\{X\}$ , the analytic germs of the corresponding Fitting loci are identified, via the local exponential isomorphism.

**Remark 5.8.** Suppose  $M$  is the complement of a hyperplane arrangement in  $\mathbb{C}^m$ , with fundamental group  $G = \pi_1(M)$ , and  $\rho = \text{id}_{\text{GL}_1(\mathbb{C})}$ . In this case, Theorem A can be deduced from results of Esnault–Schechtman–Viehweg [29] and Schechtman–Terao–Varchenko [70]. In fact, one can show that there is a combinatorially defined open neighborhood  $U$  of 0 in  $H^1(M, \mathbb{C})$  with the property that  $H^*(M, \rho\mathbb{C}) \cong H^*(H^\bullet(M, \mathbb{C}), \mu_z)$ , for all  $z \in U$ , where  $\rho = \exp(z)$ . A similar approach works as soon as  $W_1(H^1(M, \mathbb{C})) = 0$ , in particular, for complements of arrangements of hypersurfaces in projective or affine space. For details, see [22, Corollary 4.6].

The local statement from Theorem A is the best one can hope for, as shown by the following classical example.

**Example 5.9.** Let  $M$  be the complement in  $S^3$  of a tame knot  $K$ . Since  $M$  is a homology circle, it follows easily that  $M$  is a formal space; therefore, its fundamental group,  $G = \pi_1(M)$ , is a 1-formal group. Let  $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$  be the Alexander polynomial of  $K$ . It is readily seen that  $\mathcal{V}_1(G) = \{1\} \amalg \text{Zero}(\Delta)$  and  $\mathcal{R}_1(G) = \{0\}$ . Thus, if  $\Delta(t) \not\equiv 1$ , then  $\exp(\mathcal{R}_1(G)) \neq \mathcal{V}_1(G)$ .

Even though the germ of  $\mathcal{V}_1(G)$  at 1 provides no information in this case, the global structure of  $\mathcal{V}_1(G)$  is quite meaningful. For example, if  $K$  is an algebraic knot, then  $\Delta(t)$  must be product of cyclotomic polynomials, as follows from work of Brauner and Zariski from the 1920s, see [58].

**Remark 5.10.** Let  $M$  be the complement in  $\mathbb{C}^2$  of an algebraic curve, with fundamental group  $G = \pi_1(M)$ , and let  $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$  be the Alexander polynomial of the total linking cover, as defined by Libgober; see [46] for details and references. It was shown in [46] that all the roots of  $\Delta(t)$  are roots of unity. This gives restrictions on which finitely presented groups can be realized as fundamental groups of plane curve complements.



Let  $\Delta^G \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  be the multivariable Alexander polynomial of an arbitrary quasi-projective group. Starting from Theorem C(2), we prove in [25] that  $\Delta^G$  must have a single essential variable, if  $n \neq 2$ . Examples from [64] show that this new obstruction efficiently detects non-quasi-projectivity of (local) algebraic link groups. Note that all roots of the one-variable (local) Alexander polynomial  $\Delta(t)$  of algebraic links are roots of unity; see [58].

## 6. POSITION AND RESONANCE OBSTRUCTIONS

In this section, we prove Theorem C from the Introduction. The basic tool is a result of Arapura [3], which we start by recalling.

**6.1. Arapura's theorem.** First, we establish some terminology. By a *curve* we mean a smooth, connected, complex algebraic variety of dimension 1. A curve  $C$  admits a canonical compactification  $\overline{C}$ , obtained by adding a finite number of points.

Following [3, p. 590], we say a map  $f: M \rightarrow C$  from a connected, quasi-compact Kähler manifold  $M$  to a curve  $C$  is *admissible* if  $f$  is holomorphic and surjective, and has a holomorphic, surjective extension with connected fibers,  $\overline{f}: \overline{M} \rightarrow \overline{C}$ , where  $\overline{M}$  is a smooth compactification, obtained by adding divisors with normal crossings.

With these preliminaries, we can state Arapura's result [3, Proposition V.1.7], in a slightly modified form, suitable for our purposes.

**Theorem 6.2.** *Let  $M$  be a connected, quasi-compact Kähler manifold. Denote by  $\{\mathcal{V}^\alpha\}_\alpha$  the set of irreducible components of  $\mathcal{V}_1(\pi_1(M))$  containing 1. If  $\dim \mathcal{V}^\alpha > 0$ , then the following hold.*

- (1) *There is an admissible map,  $f_\alpha: M \rightarrow C_\alpha$ , where  $C_\alpha$  is a smooth curve with  $\chi(C_\alpha) < 0$ , such that*

$$\mathcal{V}^\alpha = f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)}$$

*and  $(f_\alpha)_\# : \pi_1(M) \rightarrow \pi_1(C_\alpha)$  is surjective.*

- (2) *There is an isomorphism*

$$H^1(M, f_{\alpha\rho}^* \mathbb{C}) \cong H^1(C_\alpha, \rho \mathbb{C}),$$

*for all except finitely many local systems  $\rho \in \mathbb{T}_{\pi_1(C_\alpha)}$ .*

When  $M$  is compact, similar results to Arapura's were obtained previously by Beauville [6] and Simpson [73]. The closely related construction of regular mappings from an algebraic variety  $M$  to a curve  $C$  starting with suitable differential forms on  $M$  goes back to Castelnuovo-de Franchis, see Catanese [12]. When both  $M$  and  $C$  are compact, the existence of a non-constant holomorphic map  $M \rightarrow C$  is closely related to the existence of an epimorphism  $\pi_1(M) \twoheadrightarrow \pi_1(C)$ , see Beauville [5] and Green–Lazarsfeld [34]. In the non-compact case, this phenomenon is discussed in Corollary V.1.9 from [3].

**6.3. Isotropic subspaces.** Before proceeding, we introduce some notions which will be of considerable use in the sequel. Let  $\mu: H^1 \wedge H^1 \rightarrow H^2$  be a  $\mathbb{C}$ -linear map, and  $\mathcal{R}_k(\mu) \subset H^1$  be the corresponding resonance varieties, as defined in Remark 3.17. One way to construct elements in these varieties is as follows.

**Lemma 6.4.** *Suppose  $V \subset H^1$  is a linear subspace of dimension  $k$ . Set  $i = \dim \operatorname{im}(\mu: V \wedge V \rightarrow H^2)$ . If  $i < k - 1$ , then  $V \subset \mathcal{R}_{k-i-1}(\mu) \subset \mathcal{R}_1(\mu)$ .*

*Proof.* Let  $x \in V$ , and set  $x_V^\perp = \{y \in V \mid \mu(x \wedge y) = 0\}$ . Clearly,  $\dim x_V^\perp \geq k - i$ . On the other hand,  $x_V^\perp / \mathbb{C} \cdot x \subset H^1(H^\bullet, \mu_x)$ , and so  $x \in \mathcal{R}_{k-i-1}(\mu)$ .  $\square$

Therefore, the subspaces  $V \subset H^1$  for which  $\dim \operatorname{im}(\mu: V \wedge V \rightarrow H^2)$  is small are particularly interesting. This remark gives a preliminary motivation for the following key definition.

**Definition 6.5.** Let  $\mu: \bigwedge^2 H^1 \rightarrow H^2$  be a  $\mathbb{C}$ -linear mapping, where  $\dim H^1 < \infty$ , and let  $V \subset H^1$  be a  $\mathbb{C}$ -linear subspace.

- (i)  $V$  is *0-isotropic* (or, *isotropic*) with respect to  $\mu$  if the restriction  $\mu^V: \bigwedge^2 V \rightarrow H^2$  is trivial.
- (ii)  $V$  is *1-isotropic* with respect to  $\mu$  if the restriction  $\mu^V: \bigwedge^2 V \rightarrow H^2$  has 1-dimensional image and is a non-degenerate skew-symmetric bilinear form.

**Example 6.6.** Let  $C$  be a smooth curve, and let  $\mu = \cup_C: \bigwedge^2 H^1(C, \mathbb{C}) \rightarrow H^2(C, \mathbb{C})$  be the usual cup-product map. There are two cases of interest to us.

- (i) If  $C$  is not compact, then  $H^2(C, \mathbb{C}) = 0$  and so any subspace  $V \subset H^1(C, \mathbb{C})$  is isotropic.
- (ii) If  $C$  is compact, of genus  $g \geq 1$ , then  $H^2(C, \mathbb{C}) = \mathbb{C}$  and  $H^1(C, \mathbb{C})$  is 1-isotropic.

Now let  $\mu_1: \bigwedge^2 H_1^1 \rightarrow H_1^2$  and  $\mu_2: \bigwedge^2 H_2^1 \rightarrow H_2^2$  be two  $\mathbb{C}$ -linear maps.

**Definition 6.7.** The maps  $\mu_1$  and  $\mu_2$  are *equivalent* (notation  $\mu_1 \simeq \mu_2$ ) if there exist linear isomorphisms  $\phi^1: H_1^1 \rightarrow H_2^1$  and  $\phi^2: \operatorname{im}(\mu_1) \rightarrow \operatorname{im}(\mu_2)$  such that  $\phi^2 \circ \mu_1 = \mu_2 \circ \wedge^2 \phi^1$ .

The key point of this definition is that the  $k$ -resonant varieties  $\mathcal{R}_k(\mu_1)$  and  $\mathcal{R}_k(\mu_2)$  can be identified under  $\phi^1$  when  $\mu_1 \simeq \mu_2$ . Moreover, subspaces that are either 0-isotropic or 1-isotropic with respect to  $\mu_1$  and  $\mu_2$  are matched under  $\phi^1$ .

**6.8. Proof of Theorem C(1).** We have  $\mathcal{V}^\alpha = f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)}$ , where  $f_\alpha$  is admissible and  $\chi(C_\alpha) < 0$ ; see Theorem 6.2(1). Therefore,  $\mathcal{T}^\alpha = f_\alpha^* H^1(C_\alpha, \mathbb{C})$ . If the curve  $C_\alpha$  is non-compact, the subspace  $\mathcal{T}^\alpha$  is clearly isotropic, and  $\dim \mathcal{T}^\alpha = b_1(C_\alpha) \geq 2$ . If  $C_\alpha$  is compact and  $f_\alpha^*$  is zero on  $H^2(C_\alpha, \mathbb{C})$ , we obtain the same conclusion as before. Finally, if  $C_\alpha$  is compact and  $f_\alpha^*$  is non-zero on  $H^2(C_\alpha, \mathbb{C})$ , then plainly  $\mathcal{T}^\alpha$  is 1-isotropic and  $\dim \mathcal{T}^\alpha = b_1(C_\alpha) \geq 4$ . The isotropicity property is thus established.

**6.9. Genericity obstruction.** The next three lemmas will be used in establishing the position obstruction from Theorem C(2).

**Lemma 6.10.** *Let  $X$  be a connected quasi-compact Kähler manifold,  $C$  a smooth curve and  $f: X \rightarrow C$  a non-constant holomorphic mapping. Assume that  $f$  admits a holomorphic extension  $\hat{f}: \hat{X} \rightarrow \hat{C}$ , where  $\hat{X}$  (respectively,  $\hat{C}$ ) is a smooth compactification of  $X$  (respectively,  $C$ ). Then the induced homomorphism in homology,  $f_*: H_1(X, \mathbb{Z}) \rightarrow H_1(C, \mathbb{Z})$ , has finite cokernel.*

*Proof.* Let  $Y = \text{Sing}(\hat{f})$  be the set of singular points of  $\hat{f}$ , i.e., the set of all points  $x \in \hat{X}$  such that  $d_x \hat{f} = 0$ . Then  $Y$  is a closed analytic subset of  $\hat{X}$ . Using Remmert's Theorem, we find that  $Z = \hat{f}(Y)$  is a closed analytic subset of  $\hat{C}$ , and  $\hat{f}(\hat{X}) = \hat{C}$ . By Sard's Theorem,  $Z \neq \hat{C}$ , hence  $Z$  is a finite set.

Let  $B = (\hat{C} \setminus C) \cup Z$ ; set  $C' = \hat{C} \setminus B$ , and  $\hat{X}' = \hat{X} \setminus \hat{f}^{-1}(B) = \hat{f}^{-1}(C')$ . Then the restriction  $\hat{f}': \hat{X}' \rightarrow C'$  is a locally trivial fibration; its fiber is a compact manifold, and thus has only finitely many connected components. Using the tail end of the homotopy exact sequence of this fibration, we deduce that the induced homomorphism,  $\hat{f}'_*: \pi_1(\hat{X}') \rightarrow \pi_1(C')$ , has image of finite index.

Now note that  $i \circ \hat{f}' \circ k = f \circ j$ , where  $i: C' \rightarrow C$ ,  $j: X \setminus \hat{f}^{-1}(B) \rightarrow X$ , and  $k: X \setminus \hat{f}^{-1}(B) \rightarrow \hat{X}'$  are the inclusion maps. From the above, it follows that  $f_*: H_1(X, \mathbb{Z}) \rightarrow H_1(C, \mathbb{Z})$  has image of finite index.  $\square$

Let  $\mathcal{V}^\alpha \neq \mathcal{V}^\beta$  be two positive-dimensional, irreducible components of  $\mathcal{V}_1(\pi_1(M))$  containing 1. Realize them by pull-back, via admissible maps,  $f_\alpha: M \rightarrow C_\alpha$  and  $f_\beta: M \rightarrow C_\beta$ , as in Theorem 6.2(1). We know that generically (that is, for  $t \in C_\alpha \setminus B_\alpha$ , where  $B_\alpha$  is finite) the fiber  $f_\alpha^{-1}(t)$  is smooth and irreducible.

**Lemma 6.11.** *In the above setting, there exists  $t \in C_\alpha \setminus B_\alpha$  such that the restriction of  $f_\beta$  to  $f_\alpha^{-1}(t)$  is non-constant.*

*Proof.* Assume  $f_\beta$  has constant value,  $h(t)$ , on the fiber  $f_\alpha^{-1}(t)$ , for  $t \in C_\alpha \setminus B_\alpha$ . We first claim that this implies the existence of a continuous extension,  $h: C_\alpha \rightarrow C_\beta$ , with the property that  $h \circ f_\alpha = f_\beta$ .

Indeed, let us pick an arbitrary special value,  $t_0 \in B_\alpha$ , together with a sequence of generic values,  $t_n \in C_\alpha \setminus B_\alpha$ , converging to  $t_0$ . For any  $x \in f_\alpha^{-1}(t_0)$ , note that the order at  $x$  of the holomorphic function  $f_\alpha$  is finite. Hence, we may find a sequence,  $x_n \rightarrow x$ , such that  $f_\alpha(x_n) = t_n$ . By our assumption,  $f_\beta(x) = \lim h(t_n)$ , independently of  $x$ , which proves the claim.

At the level of character tori, the fact that  $h \circ f_\alpha = f_\beta$  implies  $\mathcal{V}^\beta = f_\beta^* \mathbb{T}_{\pi_1(C_\beta)} \subset f_\alpha^* \mathbb{T}_{\pi_1(C_\alpha)} = \mathcal{V}^\alpha$ , a contradiction.  $\square$

**Lemma 6.12.** *Let  $\mathcal{V}^\alpha$  and  $\mathcal{V}^\beta$  be two distinct irreducible components of  $\mathcal{V}_1(\pi_1(M))$  containing 1. Then  $\mathcal{V}^\alpha \cap \mathcal{V}^\beta$  is finite.*

*Proof.* We may suppose that both components are positive-dimensional. Lemma 6.11 guarantees the existence of a generic fiber of  $f_\alpha$ , say  $F_\alpha$ , with the property that the restriction of  $f_\beta$  to  $F_\alpha$ , call it  $g: F_\alpha \rightarrow C_\beta$ , is non-constant. By Lemma 6.10, there exists a positive integer  $m$  with the property that

$$(6.1) \quad m \cdot H_1(C_\beta, \mathbb{Z}) \subset \text{im}(g_*).$$

We will finish the proof by showing that  $\rho^m = 1$ , for any  $\rho \in \mathcal{V}^\alpha \cap \mathcal{V}^\beta$ .

To this end, write  $\rho = \rho_\beta \circ (f_\beta)_*$ , with  $\rho_\beta \in \mathbb{T}_{\pi_1(C_\beta)}$ . For an arbitrary element  $a \in H_1(M, \mathbb{Z})$ , we have  $\rho^m(a) = \rho_\beta(m \cdot (f_\beta)_* a)$ . From (6.1), it follows that  $m \cdot (f_\beta)_* a = (f_\beta)_*(j_\alpha)_* b$ , for some  $b \in H_1(F_\alpha, \mathbb{Z})$ , where  $j_\alpha: F_\alpha \hookrightarrow M$  is the inclusion. On the

other hand, we may also write  $\rho = \rho_\alpha \circ (f_\alpha)_*$ , with  $\rho_\alpha \in \mathbb{T}_{\pi_1(C_\alpha)}$ . Hence,  $\rho^m(a) = \rho_\alpha((f_\alpha)_*(j_\alpha)_*b) = \rho_\alpha(0) = 1$ , as claimed.  $\square$

By passing to tangent spaces, Lemma 6.12 implies Theorem C(2).

**6.13. Proof of Theorem C(3).** For this property, only 1-formality is needed. See Theorem B(3).

**6.14. Proof of Theorem C(4).** This filtration-by-dimension resonance obstruction is a consequence of position obstructions from Parts (1) and (2), in the presence of 1-formality. We may assume  $k < b_1(G)$ , since otherwise  $\mathcal{R}_k(G) = \{0\}$ , and there is nothing to prove.

To prove the desired equality, we have to check first that any non-zero element  $u \in \mathcal{R}_k(G)$  belongs to some  $\mathcal{R}^\alpha$  with  $\dim \mathcal{R}^\alpha > k$ . The definition of  $\mathcal{R}_k$  guarantees the existence of elements  $v_1, \dots, v_k \in H^1(G, \mathbb{C})$  with  $v_i \cup u = 0$ , and such that  $u, v_1, \dots, v_k$  are linearly independent. Since the subspaces  $\langle u, v_i \rangle$  spanned by the pairs  $\{u, v_i\}$  are clearly contained in  $\mathcal{R}_1(G)$ , it follows that  $\langle u, v_i \rangle \subset \mathcal{R}^{\alpha_i}$ . Necessarily  $\alpha_1 = \dots = \alpha_k := \alpha$ , since otherwise property (2) would be violated. This proves  $u \in \mathcal{R}^\alpha$ , with  $\dim \mathcal{R}^\alpha > k$ . Now, if  $p(\alpha) = 1$ , then  $\dim \mathcal{R}^\alpha > k + 1$ . For otherwise, we would have  $\mathcal{R}^\alpha = u^\perp_{\mathcal{R}^\alpha}$ , which would violate the non-degeneracy property from Definition 6.5(ii). Finally, that  $\dim \mathcal{R}^\alpha > k + p(\alpha)$  implies  $\mathcal{R}^\alpha \subset \mathcal{R}_k(G)$  follows at once from Lemma 6.4.

**6.15. Fibered quasi-Kähler groups.** The next Lemma proves Theorem C(5).

**Lemma 6.16.** *Let  $M$  be a connected quasi-compact Kähler manifold. Suppose the group  $G = \pi_1(M)$  is 1-formal. The following are then equivalent.*

- (i) *There is an epimorphism,  $\varphi: G \twoheadrightarrow \mathbb{F}_r$ , onto a free group of rank  $r \geq 2$ .*
- (ii) *The resonance variety  $\mathcal{R}_1(G)$  strictly contains  $\{0\}$ .*

*Proof.* The implication (i)  $\Rightarrow$  (ii) holds in general; this may be seen by using the  $\cup_G$ -isotropic subspace  $\varphi^*H^1(\mathbb{F}_r, \mathbb{C}) \subset H^1(G, \mathbb{C})$ . If  $G$  is 1-formal and  $\mathcal{R}_1(G)$  strictly contains  $\{0\}$ , there is a positive-dimensional component  $\mathcal{V}^\alpha$ , by Theorem C(3). Applying Theorem 6.2, we obtain an admissible map,  $f: M \rightarrow C$ , onto a smooth complex curve  $C$  with  $\chi(C) < 0$ . Property (i) follows then from [3, Corollary V.1.9].  $\square$

This finishes the proof of Theorem C from the Introduction.

We close this section with a pair of examples showing that both the quasi-Kähler and the 1-formality assumptions are needed in order for Lemma 6.16 to hold.

**Example 6.17.** Consider the smooth, quasi-projective variety  $M_1$  examined by Morgan in [59, p.203] (the complex version of the Heisenberg manifold). It is well-known that  $G_1 = \pi_1(M_1)$  is a nilpotent group. Therefore, property (i) fails for  $G_1$ . Nevertheless,  $\mathcal{R}_1(G_1) = \mathbb{C}^2$ , and so property (ii) holds for  $G_1$ . In particular, the group  $G_1$  is not 1-formal.

**Example 6.18.** Let  $N_h$  be the non-orientable surface of genus  $h \geq 1$ , that is, the connected sum of  $h$  real projective planes. It is readily seen that  $N_h$  has the rational homotopy type of a wedge of  $h - 1$  circles. Hence  $N_h$  is a formal space, and so  $\pi_1(N_h)$  is a 1-formal

group. Moreover,  $\mathcal{R}_1(\pi_1(N_h)) = \mathcal{R}_1(\mathbb{F}_{h-1})$ , and so  $\mathcal{R}_1(\pi_1(N_h)) = \mathbb{C}^{h-1}$ , provided  $h \geq 3$ . Thus, property (ii) holds for all groups  $\pi_1(N_h)$  with  $h \geq 3$ .

Now suppose there is an epimorphism  $\varphi: \pi_1(N_h) \twoheadrightarrow \mathbb{F}_r$  with  $r \geq 2$ , as in (i). Then the subspace  $\varphi^*H^1(\mathbb{F}_r, \mathbb{Z}_2) \subset H^1(\pi_1(N_h), \mathbb{Z}_2)$  has dimension at least 2, and is isotropic with respect to  $\cup_{\pi_1(N_h)}$ . Hence,  $h \geq 4$ , by Poincaré duality with  $\mathbb{Z}_2$  coefficients.

Focussing on the case  $h = 3$ , we see that the group  $\pi_1(N_3)$  is 1-formal, yet the implication (ii)  $\Rightarrow$  (i) from Lemma 6.16 fails for this group. It follows that  $\pi_1(N_3)$  cannot be realized as the fundamental group of a quasi-compact Kähler manifold. Note that this assertion is *not* a consequence of Theorem C, Parts (1), (2) and (4). Indeed,  $\cup_{\pi_1(N_h)} \simeq \cup_{\mathbb{F}_{h-1}}$  (over  $\mathbb{C}$ ), while  $\mathbb{F}_{h-1} = \pi_1(\mathbb{P}^1 \setminus \{h \text{ points}\})$ , for all  $h \geq 1$ .

## 7. REGULAR MAPS ONTO CURVES AND ISOTROPIC SUBSPACES

In this section, we present a couple of useful complements to Theorem C.

**7.1. Admissible maps and isotropic subspaces.** We now consider in more detail which admissible maps  $f_\alpha: M \rightarrow C_\alpha$  may occur in Theorem 6.2.

**Proposition 7.2.** *Let  $M$  be a connected quasi-compact Kähler manifold, and let  $f: M \rightarrow C$  be an admissible map onto the smooth curve  $C$ .*

- (1) *If  $W_1(H^1(M, \mathbb{C})) = H^1(M, \mathbb{C})$ , then the curve  $C$  is either compact, or it is obtained from a compact smooth curve  $\overline{C}$  by deleting a single point.*
- (2) *If  $W_1(H^1(M, \mathbb{C})) = 0$ , then the curve  $C$  is rational. If  $\chi(C) < 0$ , then  $C$  is obtained from  $\mathbb{C}$  by deleting at least two points, and  $f^*H^1(C, \mathbb{C})$  is 0-isotropic with respect to  $\cup_M$ .*
- (3) *Assume in addition that  $\pi_1(M)$  is 1-formal. If the curve  $C$  is compact of genus at least 1, then  $f^*: H^2(C, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$  is injective, and so  $f^*H^1(C, \mathbb{C})$  is 1-isotropic with respect to  $\cup_M$ .*

*Proof.* Recall that  $f_\sharp: \pi_1(M) \rightarrow \pi_1(C)$  is surjective; hence  $f^*: H^1(C, \mathbb{C}) \rightarrow H^1(M, \mathbb{C})$  is injective.

Part (1). A quasi-compact Kähler manifold  $M$  inherits a mixed Hodge structure from each good compactification  $\overline{M}$ , by Deligne's construction in the smooth quasi-projective case [17]. Furthermore, if  $M$  is quasi-projective, this structure is unique, as shown in [17, Theorem 3.2.5(ii)].

By the admissibility condition on  $f: M \rightarrow C$ , there is a good compactification  $\overline{M}$  such that  $f$  extends to a regular morphism  $\overline{f}: \overline{M} \rightarrow \overline{C}$ . Fixing such an extension, the condition  $W_1(H^1(M, \mathbb{C})) = H^1(M, \mathbb{C})$  just means that  $j^*(H^1(\overline{M}, \mathbb{C})) = H^1(M, \mathbb{C})$ , where  $j: M \rightarrow \overline{M}$  is the inclusion. Since regular maps  $f$  which extend to good compactifications of source and target obviously preserve weight filtrations, the mixed Hodge structure on  $H^1(C, \mathbb{C})$  must be pure of weight 1, see [17]. If we write  $C = \overline{C} \setminus A$ , for some finite set  $A$ , then there is an exact Gysin sequence

$$0 \rightarrow H^1(\overline{C}, \mathbb{C}) \rightarrow H^1(C, \mathbb{C}) \rightarrow H^0(A, \mathbb{C})(-1) \rightarrow H^2(\overline{C}, \mathbb{C}) \rightarrow H^2(C, \mathbb{C}) \rightarrow 0 ,$$

see for instance [19, p. 246]. But  $H^0(A, \mathbb{C})(-1)$  is pure of weight 2, and so  $H^1(C, \mathbb{C})$  is pure of weight 1 if and only if  $|A| \leq 1$ .

Part (2). By the same argument as before, we infer in this case that  $H^1(C, \mathbb{C})$  should be pure of weight 2. The above Gysin sequence shows that  $H^1(C, \mathbb{C})$  is pure of weight 2 if and only if  $g(\overline{C}) = 0$ , i.e.,  $\overline{C} = \mathbb{P}^1$ . Finally,  $\chi(C) < 0$  implies  $|A| \geq 3$ .

Part (3). Set  $G := \pi_1(M)$ ,  $\mathbb{T}_M := \mathbb{T}_G$  and  $\mathbb{T} := \mathbb{T}_{\pi_1(C)}$ . Note that  $\dim \mathbb{T} > 0$ . Furthermore, the character torus  $\mathbb{T}$  is embedded in  $\mathbb{T}_M$ , and its Lie algebra  $T_1(\mathbb{T})$  is embedded in  $T_1(\mathbb{T}_M)$ , via the natural maps induced by  $f$ . By Theorem 6.2(2),

$$\dim H^1(M, f^*_{\rho} \mathbb{C}) = \dim H^1(C, \rho \mathbb{C}),$$

for  $\rho \in \mathbb{T}$  near 1 and different from 1, since both the surjectivity of  $f_{\sharp}$  in Part (1), and the property from Part (2) do not require the assumption  $\chi(C) < 0$ .

Applying Theorem A to both  $G$  (using our 1-formality hypothesis), and  $\pi_1(C)$  (using Example 2.5), we obtain from the above equality that

$$\dim H^1(H^{\bullet}(M, \mathbb{C}), \mu_{f^*z}) = \dim H^1(H^{\bullet}(C, \mathbb{C}), \mu_z),$$

for all  $z \in H^1(C, \mathbb{C})$  near 0 and different from 0. Moreover, for any such  $z$ , a standard calculation shows  $\dim H^1(H^{\bullet}(C, \mathbb{C}), \mu_z) = 2g - 2$ , where  $g = g(C)$ .

Now suppose  $f^*: H^2(C, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$  were not injective. Then  $f^*H^1(C, \mathbb{C})$  would be a 0-isotropic subspace of  $H^1(M, \mathbb{C})$ , containing  $f^*(z)$ . In turn, this would imply  $\dim H^1(H^{\bullet}(M, \mathbb{C}), \mu_{f^*z}) \geq 2g - 1$ , a contradiction.  $\square$

Using Theorem C, we obtain the following.

**Corollary 7.3.** *Let  $M$  be a connected quasi-compact Kähler manifold, with fundamental group  $G$ , and first resonance variety  $\mathcal{R}_1(G) = \bigcup_{\alpha} \mathcal{R}^{\alpha}$ . Assume  $b_1(G) > 0$  and  $\mathcal{R}_1(G) \neq \{0\}$ .*

- (1) *If  $M$  is compact then  $G$  is 1-formal, and each component  $\mathcal{R}^{\alpha}$  is 1-isotropic, with  $\dim \mathcal{R}^{\alpha} = 2g_{\alpha} \geq 4$ .*
- (2) *If  $W_1(H^1(M, \mathbb{C})) = 0$  then  $G$  is 1-formal, and each component  $\mathcal{R}^{\alpha}$  is 0-isotropic, with  $\dim \mathcal{R}^{\alpha} \geq 2$ .*
- (3) *If  $W_1(H^1(M, \mathbb{C})) = H^1(M, \mathbb{C})$  and  $G$  is 1-formal, then  $\dim \mathcal{R}^{\alpha} = 2g_{\alpha} \geq 2$ , for all  $\alpha$ .*

**7.4. Cohomology in degree two.** We point out the subtlety of the injectivity property from Proposition 7.2(3).

**Example 7.5.** Let  $L_g$  be the complex algebraic line bundle associated to the divisor given by a point on a projective smooth complex curve  $C_g$  of genus  $g \geq 1$ . Denote by  $M_g$  the total space of the  $\mathbb{C}^*$ -bundle associated to  $L_g$ . Clearly,  $M_g$  is a smooth, quasi-projective manifold. (For  $g = 1$ , this example was examined by Morgan in [59, p. 203].) Denote by  $f_g: M_g \rightarrow C_g$  the natural projection. This map is a locally trivial fibration, which is admissible in the sense of Arapura [3]. Since the Chern class  $c_1(L_g) \in H^2(C_g, \mathbb{Z})$  equals the fundamental class, it follows that  $f_g^*: H^2(C_g, \mathbb{C}) \rightarrow H^2(M_g, \mathbb{C})$  is the zero map. Set  $G_g = \pi_1(M_g)$ .



A straightforward analysis of the Serre spectral sequence associated to  $f_g$ , with arbitrary untwisted field coefficients, shows that  $(f_g)_*: H_1(M_g, \mathbb{Z}) \rightarrow H_1(C_g, \mathbb{Z})$  is an isomorphism, which identifies the respective character tori, to be denoted in the sequel by  $\mathbb{T}_g$ . This also implies that  $W_1(H^1(M_g, \mathbb{C})) = H^1(M_g, \mathbb{C})$ , since this property holds for the compact variety  $C_g$ .

We claim  $f_g$  induces an isomorphism

$$(7.1) \quad H^1(C_g, \rho \mathbb{C}) \xrightarrow{\cong} H^1(M_g, f_g^* \rho \mathbb{C}),$$

for all  $\rho \in \mathbb{T}_g$ . If  $\rho = 1$ , this is clear. If  $\rho \neq 1$ , then  $\text{Hom}_{\mathbb{Z}\pi_1(C_g)}(\mathbb{Z}, \rho \mathbb{C}) = 0$ , since the monodromy action of  $\pi_1(C_g)$  on  $\mathbb{Z} = H_1(\mathbb{C}^*, \mathbb{Z})$  is trivial. The claim follows from the 5-term exact sequence for twisted cohomology associated to the group extension  $1 \rightarrow \mathbb{Z} \rightarrow G_g \rightarrow \pi_1(C_g) \rightarrow 1$ ; see [39, VI.8(8.2)].

It follows that

$$(7.2) \quad \mathcal{V}_k(G_g) = \begin{cases} \mathbb{T}_g, & \text{for } 0 \leq k \leq 2g-2; \\ \{1\}, & \text{for } 2g-1 \leq k \leq 2g. \end{cases}$$

On the other hand,  $\cup_{G_g} = 0$ , since  $f_g^* = 0$  on  $H^2$ . Therefore

$$(7.3) \quad \mathcal{R}_k(G_g) = \begin{cases} T_1(\mathbb{T}_g), & \text{for } 0 \leq k \leq 2g-1; \\ \{0\}, & \text{for } k = 2g. \end{cases}$$

By inspecting (7.2) and (7.3), we see that the tangent cone formula fails for  $k = 2g-1$ . Consequently, the (quasi-projective) group  $G_g$  cannot be 1-formal. We thus see that the 1-formality hypothesis from Proposition 7.2(3) is essential for obtaining the injectivity property of  $f^*$  on  $H^2$ .

**Remark 7.6.** It is easy to show that  $f^*: H^2(C, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$  is injective when  $M$  is compact. On the other hand, consider the following genus zero example, kindly provided to us by Morihiko Saito. Take  $C = \mathbb{P}^1$  and  $M = \mathbb{P}^1 \times \mathbb{P}^1 \setminus (C_1 \cup C_2)$ , where  $C_1 = \{\infty\} \times \mathbb{P}^1$  and  $C_2$  is the diagonal in  $\mathbb{P}^1 \times \mathbb{P}^1$ . The projection of  $M$  on the first factor has as image  $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$  and affine lines as fibers; thus,  $M$  is contractible. If we take  $f: M \rightarrow \mathbb{P}^1$  to be the map induced by the second projection, we get an admissible map such that  $f^*: H^2(C, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$  is not injective.

**7.7. Twisted and Aomoto Betti numbers.** We now relate the dimensions of the cohomology groups  $H^1(H^\bullet(M, \mathbb{C}), \mu_z)$  and  $H^1(M, \rho \mathbb{C})$  corresponding to  $z \in \text{Hom}(G, \mathbb{C}) \setminus \{0\}$  and  $\rho = \exp(z) \in \text{Hom}(G, \mathbb{C}^*) \setminus \{1\}$ , to the dimension and isotropicity of the resonance component  $\mathcal{R}^\alpha$  to which  $z$  belongs.

**Proposition 7.8.** *Let  $M$  be a connected quasi-compact Kähler manifold, with fundamental group  $G$ , and first resonance variety  $\mathcal{R}_1(G) = \bigcup_\alpha \mathcal{R}^\alpha$ . If  $G$  is 1-formal, then the following hold.*

- (1) *If  $z \in \mathcal{R}^\alpha$  and  $z \neq 0$ , then  $\dim H^1(H^\bullet(M, \mathbb{C}), \mu_z) = \dim \mathcal{R}^\alpha - p(\alpha) - 1$ .*
- (2) *If  $\rho \in \exp(\mathcal{R}^\alpha)$  and  $\rho \neq 1$ , then  $\dim H^1(M, \rho \mathbb{C}) \geq \dim \mathcal{R}^\alpha - p(\alpha) - 1$ , with equality for all except finitely many local systems  $\rho$ .*

*Proof.* Part (1). Recall that  $\mathcal{R}^\alpha = f_\alpha^* H^1(C_\alpha, \mathbb{C})$ . Exactly as in the proof of Proposition 7.2(3), we infer that

$$(7.4) \quad \dim H^1(M, {}_\rho \mathbb{C}) = \dim H^1(H^\bullet(M, \mathbb{C}), \mu_z) = \dim H^1(H^\bullet(C_\alpha, \mathbb{C}), \mu_\zeta),$$

where  $z = f_\alpha^* \zeta$  and  $\rho = \exp(z)$ , for all  $z \in \mathcal{R}^\alpha$  near 0 and different from 0. Clearly

$$(7.5) \quad \dim H^1(H^\bullet(C_\alpha, \mathbb{C}), \mu_\zeta) = \dim \mathcal{R}^\alpha - p(\alpha) - 1,$$

if  $\zeta \neq 0$ . Since plainly  $\dim H^1(H^\bullet(M, \mathbb{C}), \mu_z) = \dim H^1(H^\bullet(M, \mathbb{C}), \mu_{\lambda z})$ , for all  $\lambda \in \mathbb{C}^*$ , equations (7.4) and (7.5) finish the proof of (1).

Part (2). Starting from the standard presentation of the group  $\pi_1(C_\alpha)$ , a Fox calculus computation shows that  $\dim H^1(C_\alpha, {}_{\rho'} \mathbb{C}) = \dim \mathcal{R}^\alpha - p(\alpha) - 1$ , provided  $\rho' \neq 1$ . By Theorem 6.2(2), the equality  $\dim H^1(M, {}_\rho \mathbb{C}) = \dim \mathcal{R}^\alpha - p(\alpha) - 1$  holds for all but finitely many local systems  $\rho \in \exp(\mathcal{R}^\alpha)$ . By semi-continuity, the inequality  $\dim H^1(M, {}_\rho \mathbb{C}) \geq \dim \mathcal{R}^\alpha - p(\alpha) - 1$  holds for all  $\rho \in \exp(\mathcal{R}^\alpha)$ .  $\square$

## 8. ARRANGEMENTS OF REAL PLANES

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement of planes in  $\mathbb{R}^4$ , meeting transversely at the origin. By intersecting  $\mathcal{A}$  with a 3-sphere about 0, we obtain a link  $L$  of  $n$  great circles in  $S^3$ . It is readily seen that the complement  $M$  of the arrangement deformation-retracts onto the complement of the link. Moreover, the fundamental group  $G = \pi_1(M)$  has the structure of a semidirect product of free groups,  $G = F_{n-1} \rtimes \mathbb{Z}$ , and  $M$  is a  $K(G, 1)$ . For details, see [79, 54].

**Example 8.1.** Let  $\mathcal{A} = \mathcal{A}(2134)$  be the arrangement defined in complex coordinates on  $\mathbb{R}^4 = \mathbb{C}^2$  by the half-holomorphic function  $Q(z, w) = zw(z - w)(z - 2\bar{w})$ ; see Ziegler [79, Example 2.2]. Using a computation from [54, Example 5.10], we obtain the following presentation for the fundamental group of the complement

$$G = \langle x_1, x_2, x_3, x_4 \mid (x_1, x_3^2 x_4), (x_2, x_4), (x_3, x_4) \rangle.$$

It can be seen that  $E_G = \widehat{\mathbb{L}}(x_1, x_2, x_3, x_4) / \langle\langle 2[x_1, x_3] + [x_1, x_4], [x_2, x_4], [x_3, x_4] \rangle\rangle$ , where  $\widehat{\mathbb{L}}(X)$  is the Malcev Lie algebra obtained from  $\mathbb{L}^*(X)$  by completion with respect to the degree filtration, and  $\langle\langle U \rangle\rangle$  denotes the closed Lie ideal generated by a subset  $U$ . Thus,  $G$  is 1-formal. The resonance variety  $\mathcal{R}_1(G) \subset \mathbb{C}^4$  has two components,  $\mathcal{R}^\alpha = \{x \mid x_4 = 0\}$  and  $\mathcal{R}^\beta = \{x \mid x_4 + 2x_3 = 0\}$ . The resonance obstructions from Theorem C, Parts (1), (2) and (4) are violated:

- The subspaces  $\mathcal{R}^\alpha$  and  $\mathcal{R}^\beta$  are neither 0-isotropic, nor 1-isotropic.
- $\mathcal{R}^\alpha \cap \mathcal{R}^\beta = \{x \mid x_3 = x_4 = 0\}$ , which is not equal to  $\{0\}$ .
- $\mathcal{R}_2(G) = \{x \mid x_1 = x_3 = x_4 = 0\} \cup \{x \mid x_2 = x_3 = x_4 = 0\}$ , and neither of these components equals  $\mathcal{R}^\alpha$  or  $\mathcal{R}^\beta$ .

Thus,  $G$  is not the fundamental group of any smooth quasi-projective variety.

Let  $\mathcal{A}$  be an arrangement of transverse planes in  $\mathbb{R}^4$ , with complement  $M$ . From the point of view of two classical invariants—the associated graded Lie algebra, and the Chen Lie algebra—the group  $G = \pi_1(M)$  behaves like a 1-formal group. Indeed, the

associated link  $L$  has all linking numbers equal to  $\pm 1$ , in particular, the linking graph of  $L$  is connected. Thus,  $\mathrm{gr}^*(G) \otimes \mathbb{Q} \cong \mathfrak{H}_G$  and  $\mathrm{gr}^*(G/G'') \otimes \mathbb{Q} \cong \mathfrak{H}_G/\mathfrak{H}_G''$ , as graded Lie algebras, by [52, Corollary 6.2] and [65, Theorem 10.4(f)], respectively. Nevertheless, our methods can detect non-formality, even in this delicate setting.

**Example 8.2.** Consider the arrangement  $\mathcal{A} = \mathcal{A}(31425)$  defined in complex coordinates by the function  $Q(z, w) = z(z - w)(z - 2w)(2z + 3w - 5\bar{w})(2z - w - 5\bar{w})$ ; see [55, Example 6.5]. A computation shows that  $TC_1(\mathcal{V}_2(G))$  has 9 irreducible components, while  $\mathcal{R}_2(G)$  has 10 irreducible components; see [56, Example 10.2], and [55, Example 6.5], respectively. By Theorem A, the group  $G$  is not 1-formal. Thus, the complement  $M$  cannot be a formal space, despite a claim to the contrary by Ziegler [79, p. 10].

## 9. WEDGES AND PRODUCTS

In this section, we analyze products and coproducts of groups, together with their counterparts at the level of first resonance varieties. Using our obstructions, we obtain conditions for realizability of free products of groups by quasi-compact Kähler manifolds.

**9.1. Products, coproducts, and 1-formality.** Let  $\mathbb{F}(X)$  be the free group on a finite set  $X$ , and let  $\mathbb{L}^*(X)$  be the free Lie algebra on  $X$ , over a field  $\mathbb{k}$  of characteristic 0. Denote by  $\widehat{\mathbb{L}}(X)$  the Malcev Lie algebra obtained from  $\mathbb{L}^*(X)$  by completion with respect to the degree filtration. Define the group homomorphism  $\kappa_X: \mathbb{F}(X) \rightarrow \exp(\widehat{\mathbb{L}}(X))$  by  $\kappa_X(x) = x$  for  $x \in X$ . Standard commutator calculus [45] shows that

$$(9.1) \quad \mathrm{gr}^*(\kappa_X): \mathrm{gr}^*(\mathbb{F}(X)) \otimes \mathbb{k} \xrightarrow{\cong} \mathrm{gr}_F^*(\widehat{\mathbb{L}}(X))$$

is an isomorphism. It follows from [69, Appendix A] that  $\kappa_X$  is a Malcev completion.

Now let  $G$  be a finitely presented group, with presentation  $G = \langle x_1, \dots, x_s \mid w_1, \dots, w_r \rangle$ , or, for short,  $G = \mathbb{F}(X)/\langle \mathbf{w} \rangle$ . Denote by  $\langle\langle \mathbf{w} \rangle\rangle$  the closed Lie ideal of  $\widehat{\mathbb{L}}(X)$  generated by  $\kappa_X(w_1), \dots, \kappa_X(w_r)$ , and consider the group morphism induced by  $\kappa_X$ ,

$$(9.2) \quad \kappa_G: G \rightarrow \exp(\widehat{\mathbb{L}}(X)/\langle\langle \mathbf{w} \rangle\rangle).$$

It follows from [62] that  $\kappa_G$  is a Malcev completion for  $G$ . (For the purposes of that paper, it was assumed that  $G_{\mathrm{ab}}$  had no torsion, see [62, Example 2.1]. Actually, the proof of the Malcev completion property applies verbatim in the general case, see [62, Theorem 2.2].)

**Proposition 9.2.** *If  $G_1$  and  $G_2$  are finitely presented 1-formal groups, then their coproduct  $G_1 * G_2$  and their product  $G_1 \times G_2$  are again 1-formal groups.*

*Proof.* First consider two arbitrary finitely presented groups, with presentations  $G_1 = \mathbb{F}(X)/\langle \mathbf{u} \rangle$  and  $G_2 = \mathbb{F}(Y)/\langle \mathbf{v} \rangle$ . Then  $G_1 * G_2 = \mathbb{F}(X \cup Y)/\langle \mathbf{u}, \mathbf{v} \rangle$ . It follows from (9.2) that  $E_{G_1 * G_2} = E_{G_1} \amalg E_{G_2}$ , the coproduct Malcev Lie algebra.

On the other hand,  $G_1 \times G_2 = \widehat{\mathbb{L}}(X \cup Y)/\langle \kappa_X(\mathbf{u}), \kappa_Y(\mathbf{v}), (x, y); x \in X, y \in Y \rangle$ , and so, by the same reasoning,  $E_{G_1 \times G_2} = \widehat{\mathbb{L}}(X \cup Y)/\langle \kappa_X(\mathbf{u}), \kappa_Y(\mathbf{v}), (x, y); x \in X, y \in Y \rangle$ . Using the Campbell-Hausdorff formula, we may replace each CH-group commutator  $(x, y)$  with the corresponding Lie bracket,  $[x, y]$ ; see [63, Lemma 2.5] for details. We conclude that  $E_{G_1 \times G_2} = E_{G_1} \amalg E_{G_2}$ , the product Malcev Lie algebra.

Now assume  $G_1$  and  $G_2$  are 1-formal. Hence, we may write  $E_{G_1} = \widehat{\mathbb{L}}(X')/\langle\langle \mathbf{u}' \rangle\rangle$  and  $E_{G_2} = \widehat{\mathbb{L}}(Y')/\langle\langle \mathbf{v}' \rangle\rangle$ , where the defining relations  $\mathbf{u}'$  and  $\mathbf{v}'$  are quadratic. Therefore

$$\begin{aligned} E_{G_1 * G_2} &= \widehat{\mathbb{L}}(X' \cup Y')/\langle\langle \mathbf{u}', \mathbf{v}' \rangle\rangle, \\ E_{G_1 \times G_2} &= \widehat{\mathbb{L}}(X' \cup Y')/\langle\langle \mathbf{u}', \mathbf{v}', [x', y']; x' \in X', y' \in Y' \rangle\rangle. \end{aligned}$$

Since the relations in these presentations are clearly quadratic, the 1-formality of both  $G_1 * G_2$  and  $G_1 \times G_2$  follows.  $\square$

**9.3. Products, coproducts, and resonance.** Let  $U^i, V^i$  ( $i = 1, 2$ ) be complex vector spaces, where  $U^1$  and  $V^1$  are finite-dimensional. Given two  $\mathbb{C}$ -linear maps,  $\mu_U: U^1 \wedge U^1 \rightarrow U^2$  and  $\mu_V: V^1 \wedge V^1 \rightarrow V^2$ , set  $W^i = U^i \oplus V^i$ , and define  $\mu_U * \mu_V: W^1 \wedge W^1 \rightarrow W^2$  as follows:

$$\mu_U * \mu_V|_{U^1 \wedge U^1} = \mu_U, \quad \mu_U * \mu_V|_{V^1 \wedge V^1} = \mu_V, \quad \mu_U * \mu_V|_{U^1 \wedge V^1} = 0.$$

When  $\mu_U = \cup_{G_1}$  and  $\mu_V = \cup_{G_2}$ , then clearly  $\mu_U * \mu_V = \cup_{G_1 * G_2}$ , since  $K(G_1 * G_2, 1) = K(G_1, 1) \vee K(G_2, 1)$ .

**Lemma 9.4.** *Suppose  $U^1 \neq 0$ ,  $V^1 \neq 0$ , and  $\mu_U * \mu_V$  satisfies the isotropicity resonance obstruction, i.e., each irreducible component of  $\mathcal{R}_1(\mu_U * \mu_V)$  is a  $p$ -isotropic subspace of  $W^1$ , in the sense of Definition 6.5. Then  $\mu_U = \mu_V = 0$ .*

*Proof.* Set  $\mu := \mu_U * \mu_V$ . We know  $\mathcal{R}_1(\mu) = W^1$ , by [66, Lemma 5.2]. If  $\mu \neq 0$ , then  $\mu$  is 1-isotropic, with 1-dimensional image. It follows that either  $\mu_U = 0$  or  $\mu_V = 0$ . In either case,  $\mu$  fails to be non-degenerate, a contradiction. Thus,  $\mu = 0$ , and so  $\mu_U = \mu_V = 0$ .  $\square$

Next, given  $\mu_U$  and  $\mu_V$  as above, set  $Z^1 = U^1 \oplus V^1$  and  $Z^2 = U^2 \oplus V^2 \oplus (U^1 \otimes V^1)$ , and define  $\mu_U \times \mu_V: Z^1 \wedge Z^1 \rightarrow Z^2$  as follows. As before, the restrictions of  $\mu_U \times \mu_V$  to  $U^1 \wedge U^1$  and  $V^1 \wedge V^1$  are given by  $\mu_U$  and  $\mu_V$ , respectively. On the other hand,  $\mu_U \times \mu_V(u \wedge v) = u \otimes v$ , for  $u \in U^1$  and  $v \in V^1$ . Finally, if  $\mu_U = \cup_{G_1}$  and  $\mu_V = \cup_{G_2}$ , then  $\mu_U \times \mu_V = \cup_{G_1 \times G_2}$ , since  $K(G_1 \times G_2, 1) = K(G_1, 1) \times K(G_2, 1)$ .

**Lemma 9.5.** *With notation as above,  $\mathcal{R}_1(\mu_U \times \mu_V) = \mathcal{R}_1(\mu_U) \times \{0\} \cup \{0\} \times \mathcal{R}_1(\mu_V)$ .*

*Proof.* Set  $\mu = \mu_U \times \mu_V$ . The inclusion  $\mathcal{R}_1(\mu) \supset \mathcal{R}_1(\mu_U) \times \{0\} \cup \{0\} \times \mathcal{R}_1(\mu_V)$  is obvious. To prove the other inclusion, assume  $\mathcal{R}_1(\mu) \neq 0$  (otherwise, there is nothing to prove), and pick  $0 \neq a + b \in \mathcal{R}_1(\mu)$ , with  $a \in U^1$  and  $b \in V^1$ . By definition of  $\mathcal{R}_1(\mu)$ , there is  $x + y \in U^1 \oplus V^1$  such that  $(a + b) \wedge (x + y) \neq 0$  and

$$(9.3) \quad \mu((a + b) \wedge (x + y)) = \mu_U(a \wedge x) + \mu_V(b \wedge y) + a \otimes y - x \otimes b = 0.$$

In particular,  $a \otimes y = x \otimes b$ . There are several cases to consider.

If  $a \neq 0$  and  $b \neq 0$ , we must have  $x = \lambda a$  and  $y = \lambda b$ , for some  $\lambda \in \mathbb{C}$ , and so  $(a + b) \wedge (x + y) = (a + b) \wedge \lambda(a + b) = 0$ , a contradiction.

If  $b = 0$ , then  $a \neq 0$  and (9.3) forces  $y = 0$  and  $\mu_U(a \wedge x) = 0$ . Since  $(a + b) \wedge (x + y) = a \wedge x \neq 0$ , it follows that  $a \in \mathcal{R}_1(\mu_U)$ , as needed. The other case,  $a = 0$ , leads by the same reasoning to  $b \in \mathcal{R}_1(\mu_V)$ .  $\square$

If  $G_1$  and  $G_2$  are finitely generated groups, Lemma 9.5 implies that  $\mathcal{R}_1(G_1 \times G_2) = \mathcal{R}_1(G_1) \times \{0\} \cup \{0\} \times \mathcal{R}_1(G_2)$ . An analogous formula holds for the characteristic varieties:  $\mathcal{V}_1(G_1 \times G_2) = \mathcal{V}_1(G_1) \times \{1\} \cup \{1\} \times \mathcal{V}_1(G_2)$ , see [16, Theorem 3.2].

**9.6. Quasi-projectivity of coproducts.** Here is an application of Theorem C. It is inspired by a result of M. Gromov, who proved in [35] that no non-trivial free product of groups can be realized as the fundamental group of a compact Kähler manifold. We need two lemmas.

**Lemma 9.7.** *Let  $G$  be a finitely presented, commutator relators group (that is,  $G = \mathbb{F}(X)/\langle \mathbf{w} \rangle$ , with  $X$  and  $\mathbf{w}$  finite, and  $\mathbf{w} \subset \Gamma_2 \mathbb{F}(X)$ ). Suppose  $G$  is 1-formal, and  $\cup_G = 0$ . Then  $G$  is a free group.*

*Proof.* Pick a presentation  $G = \mathbb{F}(X)/\langle \mathbf{w} \rangle$ , with all relators  $w_i$  words in the commutators  $(g, h)$ , where  $g, h \in \mathbb{F}(X)$ . We have  $E_G = \hat{\mathfrak{H}}(G)$ , by the 1-formality of  $G$ , and  $\hat{\mathfrak{H}}(G) = \mathbb{L}(X)$ , by the vanishing of  $\cup_G$ . Hence,  $E_G = \hat{\mathbb{L}}(X)$ . On the other hand, (9.2) implies  $E_G = \hat{\mathbb{L}}(X)/\langle \langle \mathbf{w} \rangle \rangle$ . We thus obtain a filtered Lie algebra isomorphism,  $\hat{\mathbb{L}}(X) \xrightarrow{\sim} \hat{\mathbb{L}}(X)/\langle \langle \mathbf{w} \rangle \rangle$ .

Taking quotients relative to the respective Malcev filtrations and comparing vector space dimensions, we see that  $\kappa_X(w_i) \in \bigcap_{k \geq 1} F_k \hat{\mathbb{L}}(X) = 0$ , for all  $i$ . A well-known result of Magnus (see [51]) says that  $\text{gr}^*(\mathbb{F}(X))$  is a torsion-free graded abelian group. We infer from (9.1) that  $w_i \in \bigcap_{k \geq 1} \Gamma_k \mathbb{F}(X)$ , for all  $i$ . Another well-known result of Magnus (see [51]) insures that  $\mathbb{F}(X)$  is residually nilpotent, i.e.,  $\bigcap_{k \geq 1} \Gamma_k \mathbb{F}(X) = 1$ . Hence,  $w_i = 1$ , for all  $i$ , and so  $G = \mathbb{F}(X)$ .  $\square$

**Lemma 9.8.** *Let  $G_1$  and  $G_2$  be finitely presented groups with non-zero first Betti number. Then  $\mathcal{V}_1(G_1 * G_2) = \mathbb{T}_{G_1 * G_2}$ .*

*Proof.* Let  $G = \langle x_1, \dots, x_s \mid w_1, \dots, w_r \rangle$  be an arbitrary finitely presented group, and let  $\rho \in \mathbb{T}_G$  be an arbitrary character. By Fox calculus, we know that  $\rho \in \mathcal{V}_1(G)$  if and only if  $b_1(G, \rho) > 0$ , where  $b_1(G, \rho) := \dim \ker d_1(\rho) - \text{rank } d_2(\rho)$ . Moreover, the linear map  $d_1(\rho): \mathbb{C}^s \rightarrow \mathbb{C}$  sends the basis element corresponding to the generator  $x_i$  to  $\rho(x_i) - 1$ , while the linear map  $d_2(\rho): \mathbb{C}^r \rightarrow \mathbb{C}^s$  is given by the evaluation at  $\rho$  of the matrix of free derivatives of the relators,  $(\frac{\partial w_j}{\partial x_i}(\rho))$ ; see Fox [32].

For  $G = G_1 * G_2$ , write  $\rho = (\rho_1, \rho_2)$ , with  $\rho_i \in \mathbb{T}_{G_i}$ . We then have  $d_j(\rho) = d_j(\rho_1) + d_j(\rho_2)$ , for  $j = 1, 2$ . Hence,  $b_1(G, \rho) = b_1(G_1, \rho_1) + b_1(G_2, \rho_2) + 1$ , if both  $\rho_1$  and  $\rho_2$  are different from 1, and otherwise  $b_1(G, \rho) = b_1(G_1, \rho_1) + b_1(G_2, \rho_2)$ . Since  $b_1(G_i, 1) = b_1(G_i) > 0$ , the claim follows.  $\square$

**Theorem 9.9.** *Let  $G_1$  and  $G_2$  be finitely presented groups with non-zero first Betti number.*

- (1) *If the coproduct  $G_1 * G_2$  is quasi-Kähler, then  $\cup_{G_1} = \cup_{G_2} = 0$ .*
- (2) *Assume moreover that  $G_1$  and  $G_2$  are 1-formal, presented by commutator relators only. Then  $G_1 * G_2$  is a quasi-Kähler group if and only if both  $G_1$  and  $G_2$  are free.*

*Proof.* Part (1). Set  $G = G_1 * G_2$ . From Lemma 9.8, we know that there is just one irreducible component of  $\mathcal{V}_1(G)$  containing 1, namely  $\mathcal{V} = \mathbb{T}_G^0$ , the component of the

identity in the character torus. Hence,  $T_1(\mathcal{V}) = H^1(G, \mathbb{C})$ . Libgober's result from [48] implies then that  $\mathcal{R}_1(G) = H^1(G, \mathbb{C})$ . If  $G$  is quasi-Kähler, Theorem C(1) may be invoked to infer that  $\cup_G$  satisfies the isotropicity resonance obstruction. The conclusion follows from Lemma 9.4.

Part (2). If  $G_1$  and  $G_2$  are free, then  $G_1 * G_2$  is also free (of finite rank), thus quasi-projective. For the converse, use Part (1) to deduce that  $\cup_{G_1} = \cup_{G_2} = 0$ , and then apply Lemma 9.7.  $\square$

Let  $\mathcal{C}$  be the class of fundamental groups of complex projective curves of non-zero genus. Each  $G \in \mathcal{C}$  is a 1-formal group, admitting a presentation with a single commutator relator, and is not free (for instance, since  $\cup_G \neq 0$ ). Proposition 9.2 and Theorem 9.9 yield the following corollary.

**Corollary 9.10.** *If  $G_1, G_2 \in \mathcal{C}$ , then  $G_1 * G_2$  is a 1-formal group, yet  $G_1 * G_2$  is not realizable as the fundamental group of a smooth, quasi-projective variety  $M$ .*

This shows that 1-formality and quasi-projectivity may exhibit contrasting behavior with respect to the coproduct operation for groups.

## 10. CONFIGURATION SPACES

Denote by  $S^{\times n}$  the  $n$ -fold cartesian product of a connected space  $S$ . Consider the *configuration space* of  $n$  distinct labeled points in  $S$ ,

$$F(S, n) = S^{\times n} \setminus \bigcup_{i < j} \Delta_{ij},$$

where  $\Delta_{ij}$  is the diagonal  $\{s \in S^{\times n} \mid s_i = s_j\}$ . The topology of configuration spaces has attracted considerable attention over the years. For  $S$  a smooth, complex projective variety, the cohomology algebra  $H^*(F(S, n), \mathbb{C})$  has been described by Totaro [76], solely in terms of  $n$  and the cohomology algebra  $H^*(S, \mathbb{C})$ .

Let  $C_g$  be a smooth compact complex curve of genus  $g$  ( $g \geq 1$ ). The fundamental group of the configuration space  $M_{g,n} := F(C_g, n)$  may be identified with  $P_{g,n}$ , the pure braid group on  $n$  strings of the underlying Riemann surface. Starting from Totaro's description, it is straightforward to check that the low-degrees cup-product map of  $P_{g,n}$  is equivalent, in the sense of Definition 6.7, to the composite

$$(10.1) \quad \mu_{g,n}: \bigwedge^2 H^1(C_g^{\times n}, \mathbb{C}) \xrightarrow{\cup_{C_g^{\times n}}} H^2(C_g^{\times n}, \mathbb{C}) \twoheadrightarrow H^2(C_g^{\times n}, \mathbb{C}) / \text{span}\{[\Delta_{ij}]\}_{i < j},$$

where  $[\Delta_{ij}] \in H^2(C_g^{\times n}, \mathbb{C})$  denotes the dual class of the diagonal  $\Delta_{ij}$ , and the second arrow is the canonical projection. It follows that the connected smooth quasi-projective complex variety  $M_{g,n}$  has the property that  $W_1(H^1(M_{g,n}, \mathbb{C})) = H^1(M_{g,n}, \mathbb{C})$ , for all  $g, n \geq 1$ .

The Malcev Lie algebra of  $P_{g,n}$  has been computed by Bezrukavnikov in [9], for all  $g, n \geq 1$ . It turns out that the groups  $P_{g,n}$  are 1-formal, for  $g > 1$  and  $n \geq 1$ , or  $g = 1$  and  $n \leq 2$ ; see [9, p. 130]. On the other hand, Bezrukavnikov also states in [9, Proposition 4.1(a)] that  $P_{1,n}$  is not 1-formal for  $n \geq 3$ , without giving an argument. With our methods, this can be easily proved.



**Example 10.1.** Let  $\{a, b\}$  be the standard basis of  $H^1(C_1, \mathbb{C}) = \mathbb{C}^2$ . Note that the cohomology algebra  $H^*(C_1^{\times n}, \mathbb{C})$  is isomorphic to  $\bigwedge^*(a_1, b_1, \dots, a_n, b_n)$ . Denote by  $(x_1, y_1, \dots, x_n, y_n)$  the coordinates of  $z \in H^1(P_{1,n}, \mathbb{C})$ . Using (10.1), it is readily seen that

$$\mathcal{R}_1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j < n \end{array} \right\}.$$

Suppose  $n \geq 3$ . Then  $\mathcal{R}_1(P_{1,n})$  is a rational normal scroll in  $\mathbb{C}^{2(n-1)}$ , see [38], [28]. In particular,  $\mathcal{R}_1(P_{1,n})$  is an irreducible, non-linear variety. From Theorem B(1), we conclude that  $P_{1,n}$  is indeed non-1-formal. This indicates that Theorem 1.3 from [37] cannot hold in the stated generality.

This family of examples also shows that both the  $\mathcal{R}_1$ -version of Arapura's result on  $\mathcal{V}_1$  from Theorem 6.2(1) and the isotropicity resonance obstruction may fail, for an arbitrary smooth quasi-projective variety  $M$ .

For  $n \leq 2$ , things are even simpler.

**Example 10.2.** It follows from (10.1) that  $\mu_{1,2}$  equals the canonical projection

$$\mu_{1,2}: \bigwedge^2(a_1, b_1, a_2, b_2) \twoheadrightarrow \bigwedge^2(a_1, b_1, a_2, b_2)/\mathbb{C} \cdot (a_1 - a_2)(b_1 - b_2).$$

It follows that  $\mathcal{R}_1(P_{1,2})$  is a 2-dimensional, 0-isotropic linear subspace of  $H^1(P_{1,2}, \mathbb{C})$ .

Consider now the smooth variety  $M'_g := M_{1,2} \times C_g$ , with  $g \geq 2$ . By Proposition 9.2, this variety has 1-formal fundamental group. It also has the property that  $W_1(H^1(M'_g, \mathbb{C})) = H^1(M'_g, \mathbb{C})$ . We infer from Lemma 9.5 that

$$\mathcal{R}_1(\pi_1(M'_g)) = \mathcal{R}_1(P_{1,2}) \times \{0\} \cup \{0\} \times H^1(C_g, \mathbb{C}),$$

where the component  $\mathcal{R}_1(P_{1,2})$  is 0-isotropic and the component  $H^1(C_g, \mathbb{C})$  is 1-isotropic. We thus see that both cases listed in Proposition 7.2(1) may actually occur.

**Remark 10.3.** Recall from Example 7.5 that the tangent cone formula may fail for quasi-projective groups, at least in the case when 1 is an isolated point of the characteristic variety. The following statement can be extracted from [48, p. 161]: “If  $M$  is a quasi-projective variety and 1 is not an isolated point of  $\mathcal{V}_1(\pi_1(M))$ , then  $TC_1(\mathcal{V}_1(\pi_1(M))) = \mathcal{R}_1(\pi_1(M))$ .” Taking  $M$  to be one of the configuration spaces  $M_{1,n}$ , with  $n \geq 3$ , shows that this statement does not hold, even when  $M$  is smooth.

Indeed, since  $P_{1,2}$  is 1-formal, we obtain from Theorem A that  $\mathcal{V}_1(P_{1,2})$  is 2-dimensional at 1. As is well-known, the natural surjection,  $P_{1,n} \twoheadrightarrow P_{1,2}$ , embeds  $\mathcal{V}_1(P_{1,2})$  into  $\mathcal{V}_1(P_{1,n})$ , for  $n \geq 2$ . Thus,  $\mathcal{V}_1(P_{1,n})$  is positive-dimensional at 1, for  $n \geq 2$ . On the other hand, it follows from Example 10.1 that  $TC_1(\mathcal{V}_1(P_{1,n}))$  is strictly contained in  $\mathcal{R}_1(P_{1,n})$ , for  $n \geq 3$ .

## 11. ARTIN GROUPS

In this section, we analyze the class of finitely generated Artin groups. Using the resonance obstructions from Theorem C, we give a complete answer to Serre's question for right-angled Artin groups, and we give a Malcev Lie algebra version of the answer for arbitrary Artin groups.

**11.1. Labeled graphs and Artin groups.** Let  $\Gamma = (\mathbf{V}, \mathbf{E}, \ell)$  be a labeled finite simplicial graph, with vertex set  $\mathbf{V}$ , edge set  $\mathbf{E} \subset \binom{\mathbf{V}}{2}$ , and labeling function  $\ell: \mathbf{E} \rightarrow \mathbb{N}_{\geq 2}$ . Finite simplicial graphs are identified in the sequel with labeled finite simplicial graphs with  $\ell(e) = 2$ , for each  $e \in \mathbf{E}$ .

**Definition 11.2.** The *Artin group* associated to the labeled graph  $\Gamma$  is the group  $G_\Gamma$  generated by the vertices  $v \in \mathbf{V}$  and with a defining relation

$$\underbrace{v w v \cdots}_{\ell(e)} = \underbrace{w v w \cdots}_{\ell(e)}$$

for each edge  $e = \{v, w\}$  in  $\mathbf{E}$ . If  $\Gamma$  is unlabeled, then  $G_\Gamma$  is called a *right-angled Artin group*, and is defined by commutation relations  $vw = wv$ , one for each edge  $\{v, w\} \in \mathbf{E}$ .

**Example 11.3.** Let  $\Gamma = (\mathbf{V}, \mathbf{E}, \ell)$  and  $\Gamma' = (\mathbf{V}', \mathbf{E}', \ell')$  be two labeled graphs. Denote by  $\Gamma \sqcup \Gamma'$  their disjoint union, and by  $\Gamma * \Gamma'$  their *join*, with vertex set  $\mathbf{V} \sqcup \mathbf{V}'$ , edge set  $\mathbf{E} \sqcup \mathbf{E}' \sqcup \{\{v, v'\} \mid v \in \mathbf{V}, v' \in \mathbf{V}'\}$ , and label 2 on each edge  $\{v, v'\}$ . Then

$$G_{\Gamma \sqcup \Gamma'} = G_\Gamma * G_{\Gamma'} \quad \text{and} \quad G_{\Gamma * \Gamma'} = G_\Gamma \times G_{\Gamma'}.$$

In particular, if  $\Gamma$  is a discrete graph, i.e.,  $\mathbf{E} = \emptyset$ , then  $G_\Gamma = \mathbb{F}_n$ , whereas if  $\Gamma$  is an (unlabeled) complete graph, i.e.,  $\mathbf{E} = \binom{\mathbf{V}}{2}$ , then  $G_\Gamma = \mathbb{Z}^n$ , where  $n = |\mathbf{V}|$ . More generally, if  $\Gamma$  is a complete multipartite graph (i.e., a finite join of discrete graphs), then  $G_\Gamma$  is a finite direct product of finitely generated free groups.

Given a graph  $\Gamma = (\mathbf{V}, \mathbf{E})$  and a subset of vertices  $\mathbf{W} \subset \mathbf{V}$ , we denote by  $\Gamma(\mathbf{W})$  the full subgraph of  $\Gamma$ , with vertex set  $\mathbf{W}$  and edge set  $\mathbf{E} \cap \binom{\mathbf{W}}{2}$ .

Let  $(S^1)^\mathbf{V}$  be the compact  $n$ -torus, where  $n = |\mathbf{V}|$ , endowed with the standard cell structure. Denote by  $K_\Gamma$  the subcomplex of  $(S^1)^\mathbf{V}$  having a  $k$ -cell for each subset  $\mathbf{W} \subset \mathbf{V}$  of size  $k$  for which  $\Gamma(\mathbf{W})$  is a complete graph. As shown by Charney–Davis [14] and Meier–VanWyk [57],  $K_\Gamma = K(G_\Gamma, 1)$ . In particular, the cup-product map  $\cup_{G_\Gamma}: H^1(G_\Gamma, \mathbb{C}) \wedge H^1(G_\Gamma, \mathbb{C}) \rightarrow H^2(G_\Gamma, \mathbb{C})$  may be identified with the linear map  $\cup_\Gamma: \mathbb{C}^\mathbf{V} \wedge \mathbb{C}^\mathbf{V} \rightarrow \mathbb{C}^\mathbf{E}$  defined by

$$(11.1) \quad v \cup_\Gamma w = \begin{cases} \pm \{v, w\}, & \text{if } \{v, w\} \in \mathbf{E}, \\ 0, & \text{otherwise,} \end{cases}$$

with signs determined by fixing an orientation of the edges of  $\Gamma$ .

**11.4. Jumping loci for right-angled Artin groups.** The resonance varieties of right-angled Artin groups were described explicitly in Theorem 5.5 from [66]. If  $\Gamma = (\mathbf{V}, \mathbf{E})$  is a graph, then

$$(11.2) \quad \mathcal{R}_1(G_\Gamma) = \bigcup_{\mathbf{W}} \mathbb{C}^\mathbf{W},$$

where the union is taken over all subsets  $\mathbf{W} \subset \mathbf{V}$  such that  $\Gamma(\mathbf{W})$  is disconnected, and maximal with respect to this property. Moreover, the decomposition (11.2) coincides with the decomposition into irreducible components of  $\mathcal{R}_1(G_\Gamma)$ .

Before proceeding to the Serre problem, we describe the characteristic variety of  $G_\Gamma$ . For  $W \subset V$ , define the subtorus  $\mathbb{T}_W \subset \mathbb{T}_{G_\Gamma} = (\mathbb{C}^*)^V$  by

$$\mathbb{T}_W = \{(t_v)_{v \in V} \in (\mathbb{C}^*)^V \mid t_v = 1 \text{ for } v \notin W\}.$$

The map  $\exp: T_1 \mathbb{T}_{G_\Gamma} \rightarrow \mathbb{T}_{G_\Gamma}$  is the componentwise exponential map  $\exp^V: \mathbb{C}^V \rightarrow (\mathbb{C}^*)^V$ ; its restriction to the subspace spanned by  $W$  is  $\exp^W: \mathbb{C}^W \rightarrow (\mathbb{C}^*)^W = \mathbb{T}_W$ .

**Proposition 11.5.** *Let  $G_\Gamma$  be the right-angled Artin group associated to the graph  $\Gamma = (V, E)$ . Then*

$$\mathcal{V}_1(G_\Gamma) = \bigcup_W \mathbb{T}_W,$$

where the union is over all subsets  $W \subset V$  such that  $\Gamma(W)$  is maximally disconnected. Moreover, this decomposition coincides with the decomposition into irreducible components of  $\mathcal{V}_1(G_\Gamma)$ .

*Proof.* The realization of  $K(G_\Gamma, 1)$  as a subcomplex  $K_\Gamma$  of the torus  $(S^1)^V$  yields a well-known resolution of the trivial  $\mathbb{Z}G_\Gamma$ -module  $\mathbb{Z}$  by finitely generated, free  $\mathbb{Z}G_\Gamma$ -modules, as the augmented,  $G_\Gamma$ -equivariant chain complex of the universal cover of  $K_\Gamma$ ,

$$\tilde{C}_\bullet(\tilde{K}_\Gamma): \cdots \longrightarrow \mathbb{Z}G_\Gamma \otimes C_k \xrightarrow{d_k} \mathbb{Z}G_\Gamma \otimes C_{k-1} \longrightarrow \cdots \xrightarrow{d_1} \mathbb{Z}G_\Gamma \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Here  $C_k$  denotes the free abelian group generated by the  $k$ -cells of  $K_\Gamma$ , and the boundary maps are given by

$$(11.3) \quad d_k(e_{v_1} \times \cdots \times e_{v_k}) = \sum_{i=1}^k (-1)^{i-1} (v_i - 1) \otimes e_{v_1} \times \cdots \times \widehat{e}_{v_i} \times \cdots \times e_{v_k},$$

where, for each  $v \in V$ , the symbol  $e_v$  denotes the 1-cell corresponding to the  $v$ -th factor of  $(S^1)^V$ .

Now let  $\rho = (t_v)_{v \in V} \in (\mathbb{C}^*)^V$  be an arbitrary character. Denoting by  $\{v^*\}_{v \in V}$  the basis of  $H^1(G_\Gamma, \mathbb{C})$  dual to the canonical basis of  $H_1(G_\Gamma, \mathbb{C})$ , define an element  $z \in \mathbb{C}^V = H^1(G_\Gamma, \mathbb{C})$  by  $z = \sum_{v \in V} (t_v - 1)v^*$ . Using (11.3), it is not difficult to check the following equality of cochain complexes

$$(11.4) \quad \text{Hom}_{\mathbb{Z}G_\Gamma}(\tilde{C}_\bullet(\tilde{K}_\Gamma), {}_\rho \mathbb{C}) = (H^\bullet(G_\Gamma, \mathbb{C}), \mu_z).$$

It follows then, directly from the definitions and using (11.4), that  $\rho \in \mathcal{V}_1(G_\Gamma)$  if and only if  $z \in \mathcal{R}_1(G_\Gamma)$ . Hence, the claimed decomposition of  $\mathcal{V}_1(G_\Gamma)$  is a direct consequence of the decomposition (11.2).  $\square$

**11.6. Serre's problem for right-angled Artin groups.** As shown by Kapovich and Millson in [41, Theorem 16.10], all Artin groups are 1-formal. This opens the way for approaching Serre's problem for Artin groups via resonance varieties, which, as noted above, were described explicitly in [66]. Using these tools, we find out precisely which right-angled Artin groups can be realized as fundamental groups of quasi-compact Kähler manifolds.

**Theorem 11.7.** *Let  $\Gamma = (V, E)$  be a finite simplicial graph, with associated right-angled Artin group  $G_\Gamma$ . The following are equivalent.*

- (i) *The group  $G_\Gamma$  is quasi-Kähler.*
- (ii) *Every positive-dimensional irreducible component  $\mathcal{R}^\alpha$  of  $\mathcal{R}_1(\cup_{G_\Gamma})$  is a  $p$ -isotropic linear subspace of  $H^1(G_\Gamma, \mathbb{C})$ , of dimension at least  $2p + 2$ , for some  $p = p(\alpha) \in \{0, 1\}$ .*
- (iii) *The graph  $\Gamma$  is complete multipartite graph.*
- (iv) *The group  $G_\Gamma$  is a finite product of finitely generated free groups.*

*Proof.* For the implication (i)  $\Rightarrow$  (ii), use the 1-formality of  $G_\Gamma$  and Theorem C.

The implication (ii)  $\Rightarrow$  (iii) is proved by induction on  $n = |V|$ . If  $\Gamma$  is complete, then  $\Gamma$  is the join of  $n$  graphs with one vertex. Otherwise, there is a subset  $W \subset V$  such that  $\Gamma(W)$  is disconnected, and maximal with respect to this property. Write  $W = W' \sqcup W''$ , with both  $W'$  and  $W''$  non-empty and with no edge connecting  $W'$  to  $W''$ . Then  $\Gamma(W) = \Gamma(W') \sqcup \Gamma(W'')$ , and so  $G_{\Gamma(W)} = G_{\Gamma(W')} * G_{\Gamma(W'')}$ . Hence,  $w' \cup_{\Gamma(W)} w'' = 0$ , for any  $w' \in W'$  and  $w'' \in W''$ . We infer from [66, Lemma 5.2] that  $\mathcal{R}_1(G_{\Gamma(W)}) = \mathbb{C}^W$ .

On the other hand, we know from (11.2) that  $\mathbb{C}^W$  is a positive-dimensional irreducible component of  $\mathcal{R}_1(G_\Gamma)$ . Our hypothesis implies that  $\mathbb{C}^W$  is either 0-isotropic or 1-isotropic with respect to  $\cup_{\Gamma(W)}$ . By Lemma 9.4,  $\cup_{\Gamma(W')} = \cup_{\Gamma(W'')} = 0$ . The cup-product formula (11.1) implies that  $\Gamma(W)$  is a discrete graph.

If  $W = V$ , we are done. Otherwise,  $V = W \sqcup W_1$ , with  $|W_1| < n$ . Since  $\Gamma(W)$  is maximally disconnected, this forces  $\{w, w_1\} \in E$ , for all  $w \in W$  and  $w_1 \in W_1$ . In other words,  $\Gamma$  is the join  $\Gamma(W) * \Gamma(W_1)$ ; thus,  $G_\Gamma = G_{\Gamma(W)} \times G_{\Gamma(W_1)}$ . By Lemma 9.5,  $\cup_{\Gamma(W_1)}$  inherits from  $\cup_\Gamma$  the isotropicity property. This completes the induction.

The implication (iii)  $\Rightarrow$  (iv) follows from the discussion in Example 11.3.

Finally, the implication (iv)  $\Rightarrow$  (i) follows by taking products and realizing free groups by the complex line with a number of points deleted.  $\square$

As is well-known, two right-angled Artin groups are isomorphic if and only if the corresponding graphs are isomorphic. Evidently, there are infinitely many graphs which are not joins of discrete graphs. Thus, implication (i)  $\Rightarrow$  (iii) from Theorem 11.7 allows us to recover, in sharper form, a result of Kapovich and Millson (Theorem 14.7 from [41]).

**Corollary 11.8.** *Among right-angled Artin groups  $G_\Gamma$ , there are infinitely many mutually non-isomorphic groups which are not isomorphic to fundamental groups of smooth, quasi-projective complex varieties.*

**11.9. A Malcev Lie algebra version of Serre's question.** Next, we describe a construction that associates to a labeled graph  $\Gamma = (V, E, \ell)$  an ordinary graph,  $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ , which we call the *odd contraction* of  $\Gamma$ . First define  $\Gamma_{\text{odd}}$  to be the unlabeled graph with vertex set  $V$  and edge set  $\{e \in E \mid \ell(e) \text{ is odd}\}$ . Then define  $\tilde{V}$  to be the set of connected components of  $\Gamma_{\text{odd}}$ , with two distinct components determining an edge  $\{c, c'\} \in \tilde{E}$  if and only if there exist vertices  $v \in c$  and  $v' \in c'$  which are connected by an edge in  $E$ .

**Example 11.10.** Let  $\Gamma$  be the complete graph on vertices  $\{1, 2, \dots, n-1\}$ , with labels  $\ell(\{i, j\}) = 2$  if  $|i - j| > 1$  and  $\ell(\{i, j\}) = 3$  if  $|i - j| = 1$ . The corresponding Artin group is the classical braid group on  $n$  strings,  $B_n$ . Since in this case  $\Gamma_{\text{odd}}$  is connected, the odd contraction  $\tilde{\Gamma}$  is the discrete graph with a single vertex.

**Lemma 11.11.** *Let  $\Gamma = (\mathbf{V}, \mathbf{E}, \ell)$  be a labeled graph, with odd contraction  $\tilde{\Gamma} = (\tilde{\mathbf{V}}, \tilde{\mathbf{E}})$ . Then the Malcev Lie algebra of  $G_\Gamma$  is filtered Lie isomorphic to the Malcev Lie algebra of  $G_{\tilde{\Gamma}}$ .*

*Proof.* The Malcev Lie algebra of  $G_\Gamma$  was computed in [41, Theorem 16.10]. It is the quotient of the free Malcev Lie algebra on  $\mathbf{V}$ ,  $\widehat{\mathbb{L}}(\mathbf{V})$ , by the closed Lie ideal generated by the differences  $u - v$ , for odd-labeled edges  $\{u, v\} \in \mathbf{E}$ , and by the brackets  $[u, v]$ , for even-labeled edges  $\{u, v\} \in \mathbf{E}$ . Plainly, this quotient is filtered Lie isomorphic to the quotient of  $\widehat{\mathbb{L}}(\tilde{\mathbf{V}})$  by the closed Lie ideal generated by the brackets  $[c, c']$ , for  $\{c, c'\} \in \tilde{\mathbf{E}}$ , which is just the Malcev Lie algebra of  $G_{\tilde{\Gamma}}$ .  $\square$

The Coxeter group associated to a labeled graph  $\Gamma = (\mathbf{V}, \mathbf{E}, \ell)$  is the quotient of the Artin group  $G_\Gamma$  by the normal subgroup generated by  $\{v^2 \mid v \in \mathbf{V}\}$ . If the Coxeter group  $W_\Gamma$  is finite, then  $G_\Gamma$  is quasi-projective. The proof of this assertion, due to Brieskorn [10], involves the following steps:  $W_\Gamma$  acts as a group of reflections in some  $\mathbb{C}^n$ ; the action is free on the complement  $M_\Gamma$  of the arrangement of reflecting hyperplanes of  $W_\Gamma$ , and  $G_\Gamma = \pi_1(M_\Gamma/W_\Gamma)$ ; finally, the quotient manifold  $M_\Gamma/W_\Gamma$  is a complex smooth quasi-projective variety.

It would be interesting to know exactly which (non-right-angled) Artin groups can be realized by smooth, quasi-projective complex varieties. We give an answer to this question, albeit only at the level of Malcev Lie algebras of the respective groups.

**Corollary 11.12.** *Let  $\Gamma$  be a labeled graph, with associated Artin group  $G_\Gamma$  and odd contraction the unlabeled graph  $\tilde{\Gamma}$ . The following are equivalent.*

- (i) *The Malcev Lie algebra of  $G_\Gamma$  is filtered Lie isomorphic to the Malcev Lie algebra of a quasi-Kähler group.*
- (ii) *The isotropicity property from Theorem 11.7(ii) is satisfied by  $\cup_{G_\Gamma}$ .*
- (iii) *The graph  $\tilde{\Gamma}$  is a complete multipartite graph.*
- (iv) *The Malcev Lie algebra of  $G_\Gamma$  is filtered Lie isomorphic to the Malcev Lie algebra of a finite product of finitely generated free groups.*

*Proof.* By Lemma 11.11, the Malcev Lie algebras of  $G_\Gamma$  and  $G_{\tilde{\Gamma}}$  are filtered isomorphic. Hence, the graded Lie algebras  $\text{gr}^*(G_\Gamma) \otimes \mathbb{C}$  and  $\text{gr}^*(G_{\tilde{\Gamma}}) \otimes \mathbb{C}$  are isomorphic.

From [74], we know that the kernel of the Lie bracket,  $\bigwedge^2 \text{gr}^1(G) \otimes \mathbb{C} \rightarrow \text{gr}^2(G) \otimes \mathbb{C}$ , is equal to  $\text{im}(\partial_G)$ , for any finitely presented group  $G$ . It follows that the cup-product maps  $\cup_{G_\Gamma}$  and  $\cup_{G_{\tilde{\Gamma}}}$  are equivalent, in the sense of Definition 6.7. Consequently,  $\cup_{G_\Gamma}$  satisfies the isotropicity resonance obstruction if and only if  $\cup_{G_{\tilde{\Gamma}}}$  does so.

With these remarks, the Corollary follows at once from Theorems 11.7 and C.  $\square$

**11.13. Kähler right-angled Artin groups.** With our methods, we may easily decide which right-angled Artin groups are Kähler groups.

**Corollary 11.14.** *For a right-angled Artin group  $G_\Gamma$ , the following are equivalent.*

- (i) *The group  $G_\Gamma$  is Kähler.*
- (ii) *The graph  $\Gamma$  is a complete graph on an even number of vertices.*
- (iii) *The group  $G_\Gamma$  is a free abelian group of even rank.*

*Proof.* Implications (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i) are clear. So suppose  $G_\Gamma$  is a Kähler group. By Theorem 11.7,  $\Gamma$  is a complete multi-partite graph  $\overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$ , and  $G_\Gamma = \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_r}$ . By Lemma 9.5, and abusing notation slightly,  $\mathcal{R}_1(G_\Gamma) = \bigcup_i \mathcal{R}_1(\mathbb{F}_{n_i})$ . Now, if there were an index  $i$  for which  $n_i > 1$ , then  $\mathcal{R}_1(\mathbb{F}_{n_i}) = \mathbb{C}^{n_i}$  would be a positive-dimensional, 0-isotropic, irreducible component of  $\mathcal{R}_1(G_\Gamma)$ , contradicting Corollary 7.3(1). Thus, we must have  $n_1 = \cdots = n_r = 1$ , and  $\Gamma = K_r$ . Moreover, since  $G_\Gamma = \mathbb{Z}^r$  is a Kähler group,  $r$  must be even.  $\square$

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